situation.[12]

These considerations for relatively low temperatures clearly do not apply to those nearer to T_c as here the effect of the microwaves is initially to increase the critical current. As yet we have no explanation for this effect.

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LASER PHOTON COUNTING DISTRIBUTIONS NEAR THRESHOLD

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We have observed the photoelectron counting distribution produced by light from a single-mode cw gas laser operating at output powers as low as twice the threshold output. In this region we are able to detect deviations from the predictions of linearized oscillator theory. Moreover we are able to fit all our observed counting distributions to one theoretical distribution containing a single parameter which varies with laser excitation. Measurements of single-mode laser counting distributions below threshold have been reported previously.[1,2] Preliminary measurements on lasers near but above threshold have also been reported.[3-5]

The laser consisted of a dc He-Ne discharge tube with Brewster-angle windows in a 15-cm cavity; the laser wavelength was 6328 Å. The axial mode separation is 1000 Mc/sec, which is larger than the full Doppler width of the Ne fluorescence line. An aperture was placed in the cavity to reduce the Q of off-axis modes and to reduce the background light from the discharge. A piezoelectric driver on one of the mirror mounts was used to tune an axial mode to the center of the fluorescence line. Under these conditions only a single mode was important for the excitations used in these experiments. The laser output was stabilized against slow drifts by means of feedback control of the discharge current; the feedback time constant was 0.03 sec.

The counting distributions were obtained using an S-20 photomultiplier, a 100-Mc/sec discriminator, a gated 100-Mc/sec scaler, and a multichannel analyzer. The number of counts n registered during a single 0.5-μsec counting period was used as an address, and a one was added to the memory register of the nth channel. The process was cycled at a 16-kc/sec rate until about 10^5 samples were obtained. This normally required less than 10 sec. The accumulated distribution was read onto punched cards for data processing. A variable attenuator in front of the photomultiplier was used to maintain an average counting rate of 2.5 per counting period independent of laser excitation. Under these conditions, dead-time and other systematic effects were found to be constant and relatively unimportant.

Theories of laser oscillators which avoid the usual linearizing assumptions have been given by Risken[6] and also by Lax and Hempstead.[7] These authors solve the Fokker-Planck equation for the amplitude probability density of the rotating-wave van der Pol oscillator. From their results for the steady-state amplitude, one can write down the probability density for the laser output intensity. Using Risken's result, in a different notation, and supplying the normalization, we have

$$P(I)=\frac{2}{\pi I_0}\frac{\exp(-|w|^2)}{1+\operatorname{erf}(w)}\exp\left[-\frac{I^2}{\pi I_0^2}+\frac{2wI}{\pi^{1/2}I_0}\right];\quad I>0. \tag{1}$$

Here I_0 is the average output intensity at threshold ($w=0$), and w is a parameter which varies from large negative to large positive values as the laser is brought from a state far below to a state far above threshold. The average output intensity $\bar{I}$ as a function of w is

$$\bar{I}/I_0=\pi^{1/2}w+\frac{\exp(-|w|^2)}{1+\operatorname{erf}(w)}. \tag{2}$$

Above threshold ($w>0$), the counting distribution corresponding to Eq. (1) may be found by the usual method[8] to be

$$p(n)=\frac{1}{\pi^{1/2}}\frac{D^n}{n!}\frac{\exp(-wD+D^2/4)}{1+\operatorname{erf}(w)}\sum_{m=0}^{n}\binom{n}{m}C^m\left[\Gamma\left(\frac{n-m+1}{2}\right)+(-1)^{n-m}\gamma\left(\frac{n-m+1}{2},|C|^2\right)\right];\quad C\geqslant 0. \tag{3}$$

Here $D=\pi^{1/2}\alpha I_0T$ and $C=w-D/2$, where α is the detector efficiency in electron counts per second per watt, and T is the counting interval; γ is the incomplete gamma function, Γ is the ordinary gamma function, and $\binom{n}{m}$ is a binomial coefficient. Below threshold (where w and C are both negative) the counting distribution differs from Eq. (3) only in that the sign of the incomplete gamma function is always negative.

Note that since I_0 depends on the laser configuration but not on the excitation, Eq. (3) gives $p(n)$ for all output powers in terms of the variation of the single parameter w. Furthermore, since w is related to the output power by Eq. (2) we actually have $p(n)$ as a function of the output intensity for a given laser.

We now compare the predictions of Eqs. (2) and (3) with experiment. Figure 1 shows a plot of the reduced factorial moment $H_2=[\langle n(n-1)\rangle/\langle n\rangle^2]-1$ vs $\bar{I}/I_0$. For a Poisson distribution $H_2=0$; for a geometric (Bose-Einstein) distribution $H_2=1$. The solid line is the theoretical result from Eqs. (2) and (3); the points were obtained from analysis of observed counting distributions. Since I_0 cannot be determined experimentally, it was used as the one parameter to be adjusted to give the best fit between theory and the experimental data above threshold. The agreement of theory and experiment is seen to be very good even at the lowest output powers at which the laser could be operated stably.

The coherence time of the intensity fluctuations at $\bar{I}/I_0=17$ was determined to be 13 μsec by measuring the single-detector noise spectrum. Thus in all cases reported the counting

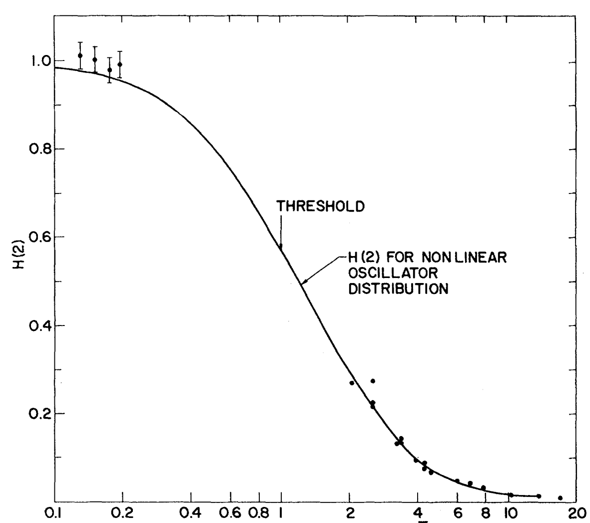

FIG. 1. The reduced factorial moment H_2 plotted against the normalized output intensity $\bar{I}/I_0$ in the region of threshold. The solid line is H_2 for the nonlinear oscillator distribution and the dots are experimental values. The counting period was 0.5 μsec, and the number of samples at each point was about 10^5.

time of 0.5 μsec was shorter than the noise correlation time.

Some data taken below threshold are also shown in Fig. 1. The error bars indicate the standard deviation for five or six runs at each output power. All the distributions observed below threshold are the geometric distribution within the experimental uncertainty. The laser stability will have to be further improved in order to follow the curve closer to threshold from below.

A typical counting distribution obtained just above threshold at $\bar{I}/I_0 = 3.25$ is shown in Fig. 2. For comparison we show both the Poisson distribution for the same $\bar{n}$ and the nonlinear oscillator distribution which gave the best fit to the data. It will be seen from Fig. 2 that the distribution of Eq. (3) is flatter than the Poisson distribution and has a longer tail; these differences are due to the intensity noise.

Well above threshold where the intensity fluctuations are relatively small, the output of the nonlinear laser oscillator can be represented as a linear superposition of an amplitude-stabilized field and a narrow-band random noise field. The counting distribution for this situation has been derived by Glauber[9] and by Lachs,[10] and in the form given by Lachs is

$$p(n) = \left\{\frac{1}{1+n_T}\left(\frac{n_T}{1+n_T}\right)^n \exp\left(\frac{-n_C}{1+n_T}\right)\right\} \times {}_1F_1\left(-n; 1; \frac{-n_C}{n_T+n_T^2}\right), \qquad (4)$$

where ${}_1F_1$ is a confluent hypergeometric function and where n_T and n_C are the average numbers of photons in the noise signal and the amplitude-stabilized signal, respectively. This distribution differs from Eq. (3) in two important respects. First, it represents the oscillator output as a linear superposition of two independent fields; second, for each output power of the laser it involves two independent parameters which are not individually related to the laser power. In obtaining the best fit of this hypergeometric distribution to the observed data, we used the relations $\bar{n} = n_T + n_C$ and $H_2 = n_T(n_T + 2n_C)/\bar{n}^2$ to determine n_T and n_C.

For purposes of comparison we have computed for our observed counting distributions the best fit by (a) a Poisson distribution, (b) the

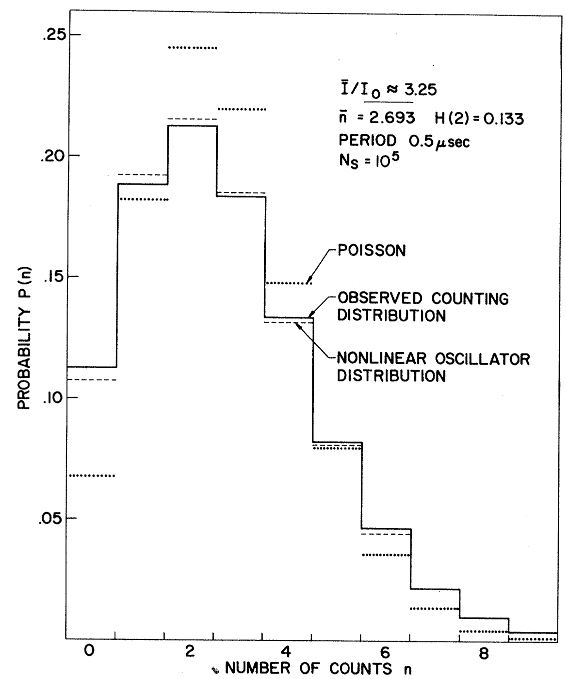

FIG. 2. Counting distribution observed just above threshold (solid line). The Poisson distribution for the same $\bar{n}$ (dotted lines) and the nonlinear oscillator distribution giving the best fit (dashed line) are also shown. For $n = 7$, 8, and 9 the nonlinear oscillator and observed distributions are coincident.

hypergeometric distribution of Eq. (4), and (c) the nonlinear oscillator distribution of Eq. (3). As a measure of the goodness of the fit we have used the quantity chi squared, defined as

$$\chi^2 = N_S \sum_n \frac{[P_n(\text{obs}) - P_n(\text{theory})]^2}{P_n(\text{theory})}, \qquad (5)$$

where the sum is over all values of n for which there are more than five samples. N_S is the total number of samples and was about 10^5 in all cases. In Table I we show the values of the chi squared obtained from six of the distributions represented in Fig. 1. The most striking result is that in every case except $\bar{I}/I_0 = 13.7$, the nonlinear oscillator distribution fits the observations significantly better than either the Poisson or the hypergeometric distributions. Even at output powers 14 times threshold there are noticeable deviations from a Poisson counting distribution. At this point, however, the quasilinear theory is quite satisfactory, as shown by the fact that Eq. (4) gives as good a fit as Eq. (3). As the laser is brought

Table I. Chi squared test of fit of counting distributions above threshold.

$\bar{I}/I_0$	Poisson	χ^2 Hypergeometric	Nonlinear oscillator
2.1	26 000	760	168
2.7	17 300	500	70
3.3	6100	245	60
4.6	1600	50	28
7.7	380	32	27
13.7	63	14	13
White light	13	...	...

closer to threshold the Poisson distribution of course becomes completely inadequate; moreover, the nonlinear oscillator distribution soon begins to show its superiority as a description of the laser output. For example, at $\bar{I}/I_0=2.7$ chi squared for the hypergeometric distribution is seven times greater than for the nonlinear oscillator distribution.[11]

It will be noted that for the while-light test run there was no significant deviation from a Poisson distribution. For the nine degrees of freedom typical of our data, the probability of observing a value of χ^2 greater than 13 due to chance is about 20%. Under the conditions of our experiment χ^2 ranged from 5 to 15 for the white-light tests. Chi squared is a statistical measure and must be expected to vary from run to run.

The fact that χ^2 for the nonlinear distribution at $\bar{I}/I_0<13.7$ is larger than can be accounted for by chance alone is due to the fact that when the laser is brought very close to threshold it becomes hypersensitive to external perturbations. The resulting intensity variations cause systematic changes in the observed counting distributions. These changes are much smaller than the great changes in the distributions which occur as threshold is approached and which are due to the nonlinear nature of the oscillator.

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