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TWO-BAND MODEL FOR BLOCH ELECTRONS IN CROSSED ELECTRIC AND MAGNETIC FIELDS

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The problem of Bloch electrons in the presence of external crossed magnetic and electric fields has recently become the subject of extensive investigations. Aronov[1] has calculated interband optical absorption in crossed fields using the one-band effective mass approximation (EMA). Peter and Hensel, Shindo, and Vrehen[2] have considered the case of degenerate bands using perturbation procedures, all based on the EMA. Recently Zak and Zawadzki,[3] using the Luttinger and Kohn treatment,[4] have shown that the validity of the one-band EMA is restricted to low values of the ratio E/H. Lax[5] indicated that to treat the problem for high values of E/H, one should start from a nonparabolic equation of relativistic type. This approach was applied by Praddaude[6] who showed that by taking into account the interaction between the two energy bands in question it is possible to obtain solutions with discrete eigenvalues for low E/H ratios and continuous eigenvalues for large E/H ratios. In this Letter we investigate this problem using essentially Kane's well-known procedure,[7] with emphasis on experimental consequences.

We calculate the matrix of the Hamiltonian for an electron in a periodic potential, with external magnetic and electric fields, in the Kohn-Luttinger representation $\chi_{nk} = \exp(i\vec{k}\cdot\vec{r}) \times u_{n0}(\vec{r})$. After returning to the coordinate representation, the set of equations for envelope functions is obtained in the following form[8]:

$$\sum_{n'}[(\vec{P}^2/2m + e\vec{E}\cdot\vec{r} + \epsilon_n - \epsilon)\delta_{n'n} + \vec{P}\cdot\vec{\pi}_{n'n}]f_{n'}(\vec{r}) = 0. \qquad (1)$$

Here $\vec{\pi}_{n'n} = (1/m)\langle u_{n'}|\vec{p}|u_n\rangle$ are determined by the interband matrix elements of momentum, ϵ_n is the energy at the bottom of the nth band (at $k=0$), $\vec{P} = \vec{p} + (e/c)\vec{A}$ is the kinetic momentum, and m is the free-electron mass. To simplify the problem we neglect the free-electron term in Eq. (1) [which is equivalent to neglecting the free mass in the effective-mass formula: $1/m_n^* = 1/m + 2\sum_{n'}|\pi_{nn'}|^2/(\epsilon_n - \epsilon_{n'})$], and consider only two spherical nondegenerate bands. Then the set of two equations can be solved by substitution. The tensor $(\vec{\pi}_{21}\vec{\pi}_{12})$ is symmetric and can be diagonalized. With the definition $1/m^* = 2|\pi_{12}|^2/\epsilon_g$, choosing the gauge $\vec{A} = [-Hy, 0, 0]$, $\vec{E} = [0, E, 0]$ in the diagonalizing coordinate system, we get

$$\left\{-\frac{\hbar^2}{2m^*}\frac{\partial^2}{\partial y^2} - \alpha y + \frac{m^*}{2}\left[\left(\frac{eH}{m^*c}\right)^2 - \frac{2e^2E^2}{m^*\epsilon_g}\right]y^2 + \frac{3}{4}\frac{\hbar^2}{2m^*}[y - (\epsilon_g + \epsilon)/eE]^{-2}\right\}\varphi(y) = \lambda\varphi(y), \qquad (2)$$

where $\alpha = \hbar\omega_0 k_x - eE(\epsilon_g + 2\epsilon)/\epsilon_g$ with $\omega_c = eH/m^*c$ the effective cyclotron frequency, and $\lambda = \epsilon(\epsilon + \epsilon_g)/\epsilon_g - \hbar^2 k_x^2/2m^* - \hbar^2 k_z^2/2m^*$. The solution in terms of $\varphi(y)$ is

$$f_1 = \exp[i(k_x x + k_z z)]\,|eEy - \epsilon_g - \epsilon|^{1/2}\varphi(y). \qquad (3)$$

The last term on the left-hand side of Eq. (2) arises from the noncommutivity of the kinetic and potential energy and is equivalent to an effective spin-orbit coupling. A term of this type occurs also in the two-band model for deep impurity levels.[9] Without the "spin-orbit" term the eigenfunctions of Eq. (2) are given by the harmonic oscillator functions if $\omega_c^2 - 2e^2E^2/m^*\epsilon_g > 0$ (with discrete eigenvalues easy to calculate in analytical form); if $\omega_c^2 - 2e^2E^2/m^*\epsilon_g = 0$ and <0 they are given by the Airy functions and parabolic cylinder functions, respectively (with continuous spectra of eigenvalues). The spin-orbit term complicates the quantitative discussion of the equation, but qualitatively these conclusions remain essentially the same.

This may be seen by examination of the effective potential in Eq. (2).

Equation (2) can be solved exactly. The solution is regular everywhere and is given by a series whose coefficients are related through a three-term recursion formula. The eigenvalues can be obtained numerically as the roots of a continued fraction.[10] We can, however, discuss the limiting cases of interest without recourse to numerical computation.

(1) $E=0$. The spin-orbit term vanishes identically and Eq. (2) reduces to the one obtained before for a magnetic field alone by Lax et al.,[11] which has been successfully applied to a number of materials.

(2) $\omega_c^2 \gg 2e^2E^2/m^*\epsilon_g$, i.e., the electric field term is negligible in comparison with the magnetic term. The spin-orbit term can be shown to be negligible, only the linear eEy term is left, and Eq. (2) has bound harmonic-oscillator solutions. The separation between adjacent levels is given in terms of $\hbar\omega_c$ and the eigenvalues are determined by the quadratic equation

$$\hbar\omega_c(n+\tfrac{1}{2}) = \epsilon(\epsilon+\epsilon_g)/\epsilon_g - \hbar^2k_z^2/2m^* - (\epsilon_g+2\epsilon/\epsilon_g)(\hbar k_x cE/H) + (\epsilon_g+2\epsilon/\epsilon_g)^2(m^*c^2E^2/2H^2). \quad (4)$$

If, in addition, we want to consider only states with low quantum numbers n in low magnetic fields, so that $\epsilon \ll \epsilon_g$, then Eq. (2) becomes the one-band effective-mass equation in crossed fields, and the eigenvalues (4) go over to those used by Aronov.[1] By use of the definition of the effective mass m^*, these requirements can be shown to be exactly equivalent to the general criteria given in Ref. 3, for the one-band EMA to be valid.

(3) $\omega_c^2 - 2e^2E^2/m^*\epsilon_g = \omega_1^2 > 0$. If the electric field term is not negligible, but not predominant either, we still have bound states and a discrete spectrum. If ω_1 does not differ much from ω_c, the spin-orbit term can again be shown to be negligible, and to a good approximation the eigenvalues are given by the relation

$$\hbar\omega_1(n+\tfrac{1}{2}) = \lambda + \tfrac{1}{2}m^*\omega_1^2y_0^2, \quad (5)$$

and the eigenfunctions $\varphi(y) = \Phi_n[(m^*\omega_1/\hbar)(y-y_0)]$ are harmonic-oscillator functions. Here $y_0 = \alpha/m^*\omega_1^2$. Thus in this region the cyclotron frequency is electric-field dependent. This is in agreement with predictions of Lifshitz and Kaganov,[12] based on a semiclassical treatment. However, for still higher electric fields when ω_1 differs appreciably from ω_c, the spin-orbit term comes into play and the eigenvalues have to be calculated numerically.

(4) $\omega_c^2 - 2e^2E^2/m^*\epsilon_g \leq 0$, i.e., the electric field term is predominant. Under this condition the spectrum of eigenvalues is essentially continuous, which is characteristic of the electric-field-type solutions. There exist also discrete eigenvalues, but they occur far above or below the bottom of the band and thus are of no interest to us.

The turning point between the magnetic and electric type of solutions can also be obtained from classical considerations. The dispersion relation between momentum and energy for two simple interacting bands is given by the simplified Kane formula as $\epsilon = -\epsilon_g/2 + [(\epsilon_g/2)^2 + \epsilon_g p^2/2m^*]^{1/2}$. It has the form of the relativistic relation with $2m_0c^2$ replaced by ϵ_g and m_0 by m^*. It is well known that the turning point between the magnetic and electric type of motion for a classical relativistic electron in crossed fields is given by $cE/H = c$.[13] Since c may be written as $(2m_0c^2/2m_0)^{1/2}$, we see that in our case the limiting velocity is $v = (\epsilon_g/2m^*)^{1/2}$. This gives the turning point at $(eH/m^*c)^2 = 2e^2E^2/m^*\epsilon_g$ in agreement with the quantum result.

The two-band model presented here accounts for the experimentally observed smooth transition from the oscillatory magnetoabsorption to the Franz-Keldysh exponential absorption edge as the electric field is increased in the cross-field configuration.[14,15] It also explains the experimental fact that at $H = 100$ kG and $E = 10^4$ V/cm, only the light-hole–electron-Landau-level transitions in Ge are observed (light holes and electrons are still in the "magnetic" region, whereas heavy holes are already in the "electric" region).

In Eq. (2) the electric and magnetic fields appear on an equal basis. This allows one to treat the magnetic field as a small perturbation if the electric field terms are predominant. This treatment is impossible when one uses the one-band EMA Hamiltonian for the crossed-field case: $H = \vec{P}^2/2m^* + e\vec{E}\cdot\vec{r}$, because here only the linear electric term is present, so that even for very small magnetic fields the bound oscillatory solutions are obtained. The experimental investigation of optical interband absorption in crossed fields[16] shows that the Franz-Keldysh exponential edge is not dramatically affected by low magnetic fields, which is in agreement with the two-band model. In a magnetic field of 100 kG and electric field 5×10^4 V/cm, the predicted decrease of cyclotron frequency in germanium amounts to 6-7%. Another optical crossed-field investigation in this material[15] seems to confirm this prediction. The approach presented is also valid for more general formulation of the Franz-Keldysh effect than that given by Tharmalingam[17] and for the description of tunneling phenomena in diodes in the presence of a magnetic field (Haering and Adams[18]).

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COHERENT ELASTIC SCATTERING AND INCOHERENT INELASTIC SCATTERING IN PROTON-NUCLEUS COLLISIONS AT HIGH ENERGY*

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Recent data on proton-nucleus scattering at 19.3 GeV/c show typical diffraction patterns.[1] For heavy nuclei they are similar to those observed from nuclear scattering at low energy, while somewhat different characteristics are shown by light nuclei. The difference may be attributed to incoherent scattering from nucleons in the nucleus which is more important for light nuclei where the transparency is larger due to the shorter paths of small-impact-parameter projectiles within the constant-density nuclear matter.

The present note uses a simple scattering model in order to extract geometrical and dynamical information from the experimental data about the target nuclei. The diffraction patterns are interpreted with the strong-absorption model (SAM)[2] which takes into account both the shadow effects of elementary plus nuclear inelastic open channels and the effect of Coulomb interaction. The large-angle cross section for light nuclei is interpreted as a free proton-nucleus scattering operating between the projectile and a small fraction of the target nucleons. This incoherent scattering is mainly an inelastic process by which the target nucleus may be transformed either into an excited state or directly into a daughter nucleus by the recoil or knock-out, respectively, of a few of its nucleons. Such events were not separated experimentally from the true elastic-scattering events.