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REMARKS ON THE CONNECTION BETWEEN EXTERNAL AND INTERNAL SYMMETRIES

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Recently a series of articles[1,2] was published, concerning the impossibility of explaining mass splitting within the context of finite-order Lie algebras, containing the Poincaré (inhomogeneous Lorentz) Lie algebra L. It is the aim of this Letter to analyze the results of the quoted articles, and to show that—lacking mathematical rigor—the author did not prove what he intended to.

To begin with, let us discuss the problem of mass differences.[2] The author claims that if the operator P^2 representing $P_\mu P^\mu$ is self-adjoint on the Hilbert space H on which a representation of a Lie algebra $E \supset L$ is defined, and if its spectrum contains a (real) eigenvalue m^2, then the (closed) eigenspace H_m of P^2 belonging to m^2 is invariant with respect to the operators representing E. We remark that all that is used about m^2 is the fact it is a point in the discrete (also[3] called point) spectrum, and it may, or may not, be isolated. Let us analyze the demonstration. We shall denote by $D(X)$ the domain of the operator X on H; obviously $H_m \subset D(P^2)$. Let e represent on H any element of E, and $H_m{}^e$ be the subspace of those h in $H_m \cap D(e)$ such that

$$eh \in \bigcap_{n=1}^{N} D(P^{2n}).$$

From the nilpotency of $P_\mu P^\mu$ in the enveloping algebra of E, we know that $ad^N P^2 = 0$. We can then consider $(P^2-m^2)^N eh$ and, since $(P^2-m^2)h = 0$ for $h \in H_m{}^e$, we have

$$(P^2-m^2)^N eh = (P^2-m^2)^{N-1}[P^2, e]h,$$

from the definition of the commutator, an expression which is well defined (notice that we cannot replace the commutator by its value before checking that the obtained expression is defined). Thus

$$(P^2-m^2)^N eh = [(ad^N P^2)e]h = 0,$$

and therefore $eH_m{}^e \subset H_m$. We can infer therefrom that the space H_m is invariant only if $H_m{}^e$ is dense in H_m, for every e in the representation of E. Now, the set

$$H^e = \{h \in H; h \in D(e), eh \in \bigcap_{n=1}^{N} D(P^{2n})\}$$

is in general a dense subspace of H (not coinciding with H), and $H_m{}^e = H_m \cap H^e$. Instead of all the H^e's, we can also consider a dense subspace D of H, on which all operators representing finite-order elements in the enveloping algebra of E are defined (this is usually the case[4]); if $H_m' = H_m \cap D$, we shall have $eH_m' \subset H_m$ for all the e's.

But, in a Hilbert space H, the intersection of a closed subspace F with a dense subspace (let us denote it by D) is not in general dense in F; for instance, we can take for F a finite-dimensional subspace, generated by some elements of H not belonging to D. Even the intersection of two dense subspaces may be $\{0\}$ (cf. reference 3, XII, 9.34).

Thus, we see that the claimed result is not proved; moreover, it is very likely wrong in the above formulation.

Let us now analyze the assumptions (cf. reference 1) on which this "theorem" is based. Let us grant the self-adjointness of P^2; the "discreteness" of the eigenvalue of P^2, an eigenvector of which represents a particle, is not used in the "proof," and therefore the distinction made afterwards between a bump and a "discrete" eigenvalue is irrelevant. As for the first assumption, it is usually admitted that two particles belonging to the same physical multiplet belong to the same irreducible representation of the internal algebra, and this does not imply that they belong to the same irreducible representation of the combined algebra.

In addition to this, one should keep in mind the following possible realizations of the link between masses and the operator P^2: One may consider expectation values, or a density in the spectrum, or use representations in spaces more general than Hilbert space.

The quoted papers[1] contain in addition some structural considerations, based on the Levi-Malcev theorem. The problem tackled being very general, it is difficult to obtain but quite superficial results. [On the contrary, with proper minimality conditions, the present authors[5] obtained, among other results, restrictions on internal symmetries, excluding compact ones.] Moreover, the results obtained —which could have been easily refined by use of deeper properties (such as the Iwasawa decomposition[6])—are, even when given unwieldy proofs, immediate, if not already known. For instance, from the construction of semidirect products,[7] and because the derivation algebra of the translation (real) Lie algebra P is $GL(4,R)$, whose only semisimple subalgebras[8] that contain the (homogeneous) Lorentz Lie algebra M realized as $SO(3,1)$ (in order to get the commutation relations of L) are $SL(4,R)$ and M, we get that if $E\supset L$ and P is an ideal in E, any Levi factor of E is isomorphic to $G=G_0\oplus G_\gamma$, where $G_0=SL(4,R)$ (complex type A_3) or M (type $A_1\oplus A_1$). This is "theorem D" of reference 1, with the exception that G_0 cannot be of type B_2, as incorrectly stated there—no four-dimensional representation of a real (noncompact) form of B_2 gives $D(\frac{1}{2},\frac{1}{2})$ on M^-. The same considerations applied to the one-dimensional Lie algebra give "theorem A" of reference 1 (while, e.g., "theorem B" or the appendix are special cases of known results). Moreover, the known fact that the conformal (simple) Lie algebra $SO(4,2)$ contains L may be of interest.[9]

Let us now consider the use made of the so-called redefinitions, e.g., in connection with the "generalization" of McGlinn's result. We shall assume that E is the direct sum of vector spaces $M+P+T$ (T: internal algebra), and consider a Levi factor G of E such that $M\subset G$. The "redefinition" of T that permits the (very strong) assumption $G=T+M$, and is in fact realized by means of a special automorphism,[7] is not a trivial mathematical operation, as we can see by the following example[5]: Take $T=SO(4,1)$, E the semidirect product of T ($=G$ here) by a 10-dimensional commutative ideal, defined by the adjoint representation; then we have $E=M+P+T=L+T\neq L\oplus T$ (M being "translated" of $SO(3,1)\subset SO(4,1)$ by a special automorphism); moreover, we can correct (nontrivially) $P_\mu P^\mu$ so as to obtain a "Casimir operator" of E. Thus redefinitions are not trivial mathematically. Notice also that the formulas (8.8) of reference 1 do not give the commutation relations of M^- and thus the discussion made about them is meaningless. As for the so-called "extended Han theorem," the Poincaré-Birkhoff-Witt theorem seems irrelevant, as $\omega^2=W_\mu W^\mu/P_\mu P^\mu$ is not defined in the enveloping algebra of L.

As to the physical point of view of the redefinition problem, the authors think that it is more a question of a good choice of a particular example than a matter of opinion. Moreover, the choice of a bad example does, of course, not imply that redefinitions are physically trivial.

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NEW DETERMINATION OF THE PION-NUCLEON COUPLING CONSTANT AND s-WAVE SCATTERING LENGTHS*

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The most accurate determination of the pion-nucleon coupling constant published to date is that of Woolcock[1] using the dispersion relation for the invariant amplitude B_+ in the forward direction. (The subscript + refers to π^+p elastic scattering.) This method gave

$$f^2 = 0.081 \pm 0.003.$$

Although we were satisfied that the method was reliable and that the quoted error was realistic, we thought that a more accurate result could now be obtained by a more direct method. This is simply to re-evaluate the familiar dispersion relations for the $\pi^\pm p$ elastic-scattering amplitudes in the forward direction, taking account of the large amount of experimental data now available. This calculation also yields the s-wave scattering lengths. Since the results are interesting (and surprising), we give here a brief account of the method used, and of the values obtained for the constants.

Instead of writing the dispersion relations in the usual form, first written down by Goldberger, Miyazawa, and Oehme,[2] we take advantage of the sum rule discussed in the same paper to eliminate one of the scattering-length combinations. The most convenient form for evaluation is to write $D_\pm$, the real parts of the $\pi^\pm p$ forward-scattering amplitudes in the laboratory system, in the form

$$D_\pm(\omega_L) = A \mp \frac{2f^2}{\omega_L \mp 1/2M} + \frac{\omega_L}{4\pi^2} P\int_0^\infty d\omega_L' \times \frac{q_L'}{\omega_L'}\left[\frac{\sigma_\pm(\omega_L')}{\omega_L'-\omega_L} - \frac{\sigma_\pm(\omega_L')}{\omega_L'+\omega_L}\right]. \quad (1)$$

$\hbar$, c, and μ, the charged pion mass, are chosen as basic units; q_L, ω_L are the momentum and total energy, respectively, of the incident pion in the laboratory system; and $M = M_p/[1-(M_n^2-M_p^2)]$, where M_n and M_p are the neutron and proton masses, respectively. $\sigma_\pm$ are the total cross sections for $\pi^\pm p$ scattering. The constants A and f^2 are to be determined.

Using the language of charge independence, if a_1 and a_3 are the usual s-wave pion-nucleon scattering lengths for $I=\frac{1}{2}$ and $\frac{3}{2}$, respectively, we have from (1)

$$D_+(1)+D_-(1) = \left(1+\frac{1}{M_p}\right)\frac{2}{3}(a_1+2a_3) = 2A - \frac{1}{M}\frac{2f^2}{1-1/4M^2} + \frac{1}{2\pi^2}\int_1^\infty \frac{d\omega_L'}{\omega_L' q_L'} \times[\sigma_+(\omega_L')+\sigma_-(\omega_L')], \quad (2)$$

$$D_-(1)-D_+(1) = \left(1+\frac{1}{M_p}\right)\frac{2}{3}(a_1-a_3) = \frac{4f^2}{1-1/4M^2} + \frac{1}{2\pi^2}\int_1^\infty \frac{d\omega_L'}{q_L'} \times[\sigma_-(\omega_L')-\sigma_+(\omega_L')]. \quad (3)$$

Thus, having determined A and f, evaluation of two further integrals gives (a_1+2a_3) and (a_1-a_3).

The values of $\sigma_\pm$ required to evaluate the dispersion integrals in Eqs. (1), (2), and (3) were obtained as follows: Up to 400 MeV, the phase shift α_{33} for the $I=\frac{3}{2}$ $p_{3/2}$ state was parametrized using the form

$$q^3 \cot\alpha_{33} = \sum_{n=0}^{n} a_n q^{2n},$$