CRITICAL STATE OF THE SUPERCONDUCTING SURFACE SHEATH*

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When a long superconducting cylinder is placed into an axial magnetic field whose value is between the upper critical field H_{c2} and the surface nucleation field H_{c3}, then the surface is in the superconducting state, and the core of the cylinder is in the normal state provided the thickness of the surface sheath Δ is small compared to the radius of the cylinder R. Although the metal is singly connected, the superconducting surface is multiply connected. By analogy with a superconducting ring, one could imagine that a persistent current could be induced in the surface sheath provided the surface sheath is able to carry a total current. Such a current could easily change the magnetic properties of a superconducting cylinder.

We shall show that such a current can indeed be induced in the surface sheath and that its magnetic moment can be either diamagnetic or paramagnetic according to whether the magnetic field is increased or decreased, respectively. It is found that these currents are size dependent and have an appreciable effect on the magnetization for Ginzburg-Landau κ values of order unity and for applied magnetic fields close to H_{c2}. It is also shown for a cylinder that the induced magnetization per unit volume is always considerably larger than the inherent diamagnetic magnetization per unit volume of the surface sheath (in its lowest energy state), although the induced surface current is always considerably smaller than the inherent surface currents provided R is very large compared to the penetration depth λ.

The free-energy difference between the superconducting state and the normal state in an applied magnetic field is, in the usual normalized Ginzburg-Landau[1] notation,

$$F_S-F_N=\oint dV\left\{(\vec{H}-\vec{H}_0)^2-|\psi|^2+\frac{|\psi|^4}{2}+\left|\frac{i}{\kappa}\vec{\nabla}\psi+\vec{A}\psi\right|^2\right\}. \tag{1}$$

$\vec{H}_0$ is the applied magnetic field, $\vec{H}(=\mathrm{curl}\,\vec{A})$ is the internal magnetic field, and the integral extends over the total volume of the superconductor. The free-energy difference is a function of the order parameter ψ and the vector potential $\vec{A}$. When we are searching for the lowest energy state we vary Eq. (1) with respect to the functions ψ and $\vec{A}$ which leads to the first and second Ginzburg-Landau equations.

When a persistent current is induced in the surface sheath by changing the applied magnetic field, the sheath is not in the lowest energy state but in an excited stable state. We now define the maximum persistent current which the sheath may carry as the critical current and the critical state as that state for which F_S-F_N in Eq. (1) is zero. The right-hand side of Eq. (1) is a function of two independently variable functions ψ and $\vec{A}$. We make the assumption that the free-energy difference Eq. (1) is still minimized with respect to ψ but not any longer with respect to $\vec{A}$. This means that the first Ginzburg-Landau equation applies,

$$[(i/\kappa)\vec{\nabla}+\vec{A}]^2\psi-\psi+|\psi|^2\psi=0. \tag{2}$$

The appropriate functions $\vec{A}$ and ψ are determined from Eq. (2) and Eq. (1) with $F_S-F_N=0$. We have now two unknown functions and two equations and our critical state is determined, at least in principle. Substitution of Eq. (2) into Eq. (1) leads (with $F_S-F_N=0$) to

$$\oint dV\{(\vec{H}-\vec{H}_0)^2-\tfrac{1}{2}|\psi|^4\}=0. \tag{3}$$

Let us consider a very long cylinder in an applied magnetic field parallel to the axis of the cylinder. The magnetic field is between H_{c2} and H_{c3} so that superconductivity exists near the surface of the cylinder but is quenched near the center of the cylinder ($R\gg\Delta$). We assume that ψ is of the following form (cylindrical coordinates):

$$\psi=e^{ib\varphi}F(\rho), \tag{4}$$

where b is an integer (number of enclosed fluxoids) and $F(\rho)$ some function of ρ. The symmetry axis of the cylinder is the z direction ($\rho=0$). The vector potential is assumed to have the following form:

$$\vec{A}=(0;A_\varphi;0), \tag{5}$$

$$A_\varphi=\tfrac{1}{2}H_0\rho+a(\rho)/\rho, \tag{6}$$

where $a(\rho)$ is some unknown function of ρ. From the definition of the flux,

$$\Phi = \oint\!\!\oint \vec{H}\cdot d\vec{S} = \oint \vec{A}\cdot d\vec{l}, \qquad (7)$$

it follows that $a(\rho)$ must be zero when $\rho = 0$.

It is our objective to calculate the magnetization per unit volume for a very long cylinder when the critical current is flowing in the surface sheath. From Eq. (6) it follows that

$$H_z = H_0 + \frac{1}{\rho}\frac{da}{d\rho}, \qquad (8)$$

and from Maxwell's first equation

$$\frac{4\pi}{c} j_\varphi(\rho) = \frac{d}{d\rho}\left(\frac{1}{\rho}\frac{da}{d\rho}\right). \qquad (9)$$

From Eq. (8) and the definition of the magnetization per unit volume we obtain

$$4\pi M = \frac{2}{R^2}\int_0^R (H_z - H_0)\rho d\rho = \frac{2}{R^2} a(R). \qquad (10)$$

In the lowest energy state[2,3] (when no current is induced in the surface sheath) two currents (per unit length of the cylinder) J_{S1} and J_{S2} of equal magnitude but opposite direction flow near the surface. We associate with J_{S1} and J_{S2} a magnetization per unit volume $4\pi M_0 = 2a_0(R)/R^2$ and with the induced critical current (per unit length of the cylinder) J_c a magnetization per unit volume $4\pi M_1 = 2a_1(R)/R^2$. We shall show below that $M_1 \gg M_0$ when $R \gg \lambda$ and there $a_1(R) \gg a_0(R)$. We introduce the notation $a(\rho) = a_0(\rho) + a_1(\rho)$ and $j_\varphi(\rho) = j_0(\rho) + j_1(\rho)$, where $j_0(\rho)$ is the current density in the lowest energy state and $j_1(\rho)$ is the current density of the induced critical current. We now substitute Eqs. (4) and (8) into Eq. (3) and obtain

$$\int_0^R \rho\left[\left(\frac{1}{\rho}\frac{da}{d\rho}\right)^2 - \tfrac{1}{2}F^4\right] d\rho = 0. \qquad (3a)$$

One solution of Eq. (3a) is that the integrand is zero. It can be seen readily that this solution is physically unrealistic for the following reasons: The order parameter is zero in the center of the cylinder and nonzero only near the surface. When a current is induced near the surface, the magnetic field near the center of the cylinder must be different from the applied field and hence $(1/\rho)(da/d\rho) \neq 0$ for $\rho = 0$. At the surface $(1/\rho)(da/d\rho) = 0$. The boundary values of $(1/\rho)(da/d\rho)$ are just opposite from those of F^2 and therefore the integrand cannot be zero.

Equation (3a) is integrated by parts with the boundary condition $a(0) = 0$ and $H_z(R) = H_0$, and one obtains

$$\frac{4\pi}{c}\int_0^R (a_0 + a_1)(j_0 + j_1) d\rho = \tfrac{1}{2}\int_0^R \rho F^4 d\rho. \qquad (3b)$$

Exact solutions of Eqs. (2) and (3b) are difficult to obtain without a computer. However, it is possible to find a very good approximation to our problem which not only shows the essentail features of the critical state but is also in good quantitative agreement with recent experiments.[4] It is easy to see that within the surface sheath $a_1(\rho) \gg a_0(\rho)$ because it was assumed previously (and will be proved below) that $a_1(R) \gg a_0(R)$. Furthermore, $a_0(\rho)/\rho$ changes from $a_0(R)/R$ to zero within the thickness of the surface sneath[2] but $a_1(\rho)/\rho$ becomes zero only at $\rho = 0$ and increases linearly with ρ over most of the cylinder. We therefore can neglect $a_0(\rho)$ with respect to $a_1(\rho)$ near the surface of the cylinder in Eq. (3b). Further, it will be shown below that

$$|J_c| = |\int_0^R j_1(\rho) d\rho| \ll |J_{S1}|. \qquad (11)$$

This implies that $da_1/d\rho$ is very small compared to $da_0/d\rho$ over most of the surface sheath. At $\rho = R$ the derivates of a_1 and a_0 with respect to ρ become zero because $H_z(R) = H_0$. Therefore a_1 must be a very slowly varying function over the surface sheath compared to a_0, and for the present considerations we shall assume that $a_1(\rho)$ is approximately a constant over the thickness of the surface sheath. Furthermore, it was shown[2,3] that

$$\int_0^R j_0(\rho) d\rho = 0$$

and, therefore, the integral on the left-hand side of Eq. (3b) can be written as $\alpha a_1(R) J_c$, where α is a parameter of order unity.

To first approximation, $a_1(\rho)/\rho$ will have no effect on the solution of ψ because $a_1(\rho)/\rho$ is also approximately a constant with respect to $a_0(\rho)/\rho$ over the surface sheath. The vector potential $\vec{A}$ is shifted by approximately a constant amount over the distance where ψ is nonzero in Eq. (2), and since Eq. (2) is gauge invariant this solution of ψ will be very close to that of the surface sheath in its lowest energy state.[2] ψ will depend approximately only on κ and H_0/H_{c2} and not on J_c since $|J_c| \ll |J_{S1}|$.

With the following definitions:

$$\Delta = \frac{1}{F^2(R)} \int_0^R F^2(\rho) d\rho, \qquad (12)$$

$$\beta = \int_0^R F^4(\rho) d\rho / \left(\int_0^R F^2(\rho) d\rho\right)^2, \qquad (13)$$

the right-hand side of Eq. (3b) is approximately $\beta R \Delta^2 F^4(R)/2$, where β is of order unity. For simplicity we define the parameter $\eta = (\beta/\alpha)^{1/2}$, where η is also of order unity. From Eq. (3b) and Eq. (10) the magnetization per unit volume of a cylinder of radius R due to the critical current J_c in the surface sheath is

$$4\pi M_1 = \pm \eta \frac{\Delta}{\sqrt{R}} F^2(R). \qquad (14)$$

In real units [unnormalized except $F^2(R)$] Eq. (14) becomes

$$4\pi M_1 = \pm \eta \frac{H_c}{\kappa} \left(\frac{2\lambda}{R}\right)^{1/2} \frac{\Delta}{\xi} F^2(R). \qquad (14a)$$

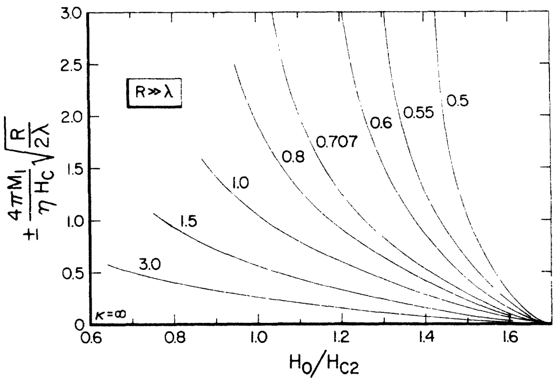

FIG. 1. The quantity $(\Delta/\xi)[F^2(R)/\kappa]$ is shown as a function of H_0/H_{c2} for various κ values as calculated from reference 2 for a semi-infinite superconducting half-space. These results are applicable to a cylinder when the radius $R \gg \Delta$ (thickness of superconducting sheath). The induced magnetization per unit volume and the critical current per unit length are related to the above quantity via Eq. (14a) and are size dependent. Below H_{c2} the solutions are extended 40% into the mixed state for type II superconductors without taking the bulk solution into account.

When the sheath is in the critical state, it follows from Eq. (14a) that the magnetization per unit volume can be either diamagnetic or paramagnetic. The sign of the magnetic moment depends on the direction in which the current is induced according to Lenz's law. Various materials with the same value of H_c but different κ values will have a magnetization per unit volume in the critical state which is approximately inversely proportional to κ (for the same value of H_0/H_{c2}). It should be noted that Δ/ξ and $F(R)$ are also functions of κ and H_0/H_{c2}, and these will change the simple $1/\kappa$ dependence of the magnetization. The values of Δ/ξ and $F(R)$ may be taken from reference 2 and Fig. 1 shows the quantity $(\Delta/\xi)[F^2(R)/\kappa]$ as a function of H_0/H_{c2} for various κ values. The magnetization per unit volume as well as the induced critical current are size dependent and are inversely proportional to the square root of the radius of the cylinder.

For a cylinder of radius R, the magnetization per unit volume of the surface sheath when in the critical state M_1 is readily compared with the magnetization per unit volume of the surface sheath in its lowest energy state[2] M_0. One finds[2]

$$\frac{M_1}{M_0} = \pm \frac{\eta}{2\kappa} \frac{H_{c2}}{H_0} \left(\frac{R}{\lambda}\right)^{1/2} \frac{\Delta}{\xi} \frac{F^2(R)}{\mu a(\infty)}. \qquad (15)$$

From reference 2 one finds that $F^2(R)/\kappa\mu a(\infty)$ is always larger than unity and, therefore, $M_1/M_0 \gg 1$ because η, H_{c2}/H_0, and Δ/ξ are of order unity and $R \gg \lambda$. This justifies our assumption that $a_1(R) \gg a_0(R)$. For example, for $\kappa = 1$, $R = 0.1$ cm, $H_c = 500$ G, and $H_{c2}/H_0 = 1$, we obtain $4\pi M_1 = 6.3$ G and $M_1/M_2 = 1.65 \times 10^2$ (with $\eta = 1$).

When the superconducting surface sheath is in the lowest energy state ($J_c = 0$), two supercurrents J_{S1} and J_{S2} of equal magnitude but opposite direction flow near the surface.[2,3] When one compares $|J_{S1}|$ from reference 3 with the critical current $(4\pi/c)J_c \approx 4\pi M_1$ from Eq. (14a), one obtains

$$\left|\frac{J_c}{J_{S1}}\right| \approx \eta \frac{2}{\kappa} \frac{\Delta}{\xi} \left(\frac{\lambda}{R} \frac{H_0}{H_{c2}}\right)^{1/2} [1 - \tfrac{1}{2}F^2(R)]^{-1/2}. \qquad (16)$$

The critical current is small compared to the inherent surface currents, because η, Δ/ξ, H_0/H_{c2}, and $[1-\frac{1}{2}F^2(R)]$ are all of order unity and $\lambda \ll R$. Therefore, the induced current is only a small perturbation on the inherent surface currents and this is the justification for using the unperturbed order parameter [calculated from Eq. (2) with $J_c = 0$] in Eq. (14a).

The possible existence of persistent surface currents was previously suggested by Livingston and Schadler[5] in relation to the work of

Chiou, Connell, and Seraphim,[6] and by one of the authors.[3] Chiou, Connell, and Seraphim[6] have observed for In−3.6 at.% Pb alloys (type I superconductor; $0.417<\kappa<0.707$) a tail in the magnetization curve which is diamagnetic for increasing magnetic fields and paramagnetic for decreasing magnetic fields. This is most likely the superconducting sheath in the critical state although Chiou, Connell, and Seraphim[6] explained their results at that time as "the usual filamentary superconductivity in high magnetic fields." The hysteresis of the magnetization near H_c is due to the critical surface current. The bulk of the metal goes into the diamagnetic state at an external field which is smaller than H_c because the internal field is enhanced due to the contribution of the critical sheath current when the external magnetic field is decreased. Unfortunately, no quantitative comparison can be made because neither the sample shape nor its size are given.

Abrikosov[7] has calculated the approximate maximum transport current which can be passed through the superconducting surface sheath of a semi-infinite half space for $\kappa \gg 1$ when the current is parallel or perpendicular to the magnetic field. He obtains a critical transport current which is by one to two orders of magnitude larger than what has been actually observed.[8] We believe that this discrepancy arises mainly because in Abrikosov's calculation the magnetic field energy, which is proportional to the volume of the sample, was neglected. Recent numerical calculations by Park[9] have the same limitations.

In calculating the critical current J_c we have assumed in Eq. (3) that the sample is in thermodynamic equilibrium ($F_S-F_N=0$). Stable current states exist[4] also for $F_S-F_N<0$ and metastable states may also exist for $F_S-F_N>0$. The latter are likely to occur when flux pinning plays a dominant role. Experiments[4] with reasonably good surfaces show, however, that the thermodynamic equilibrium state ($F_S-F_N=0$) is favored [maximum stable current; Eq. (14a)], probably because the superconducting surface is in physical contact with the normal core of the cylinder. Equation (3) (with $F_S-F_N\neq 0$) can be also solved (the same way as above) when F_S-F_N is an arbitrary constant.

A comprehensive account of the exact numerical work on the critical state of the superconducting surface sheath will be published elsewhere.

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