trary units, the following values[12,13]:

$$|A(\pi^+\rho^-, {}^3S)| = 2.94 \pm 0.15,$$
$$|A(K^0\bar{K}^{*0}, {}^3S)| = 1.23 \pm 0.13,$$
$$|A(K^+K^{*-}, {}^3S)| = 1.11 \pm 0.11. \quad (16)$$

The sum rule

$$A(\pi^+\rho^-, {}^3S) - A(K^0\bar{K}^{*0}, {}^3S) - A(K^+K^{*-}, {}^3S) = 0$$

seems well satisfied with experiment to within 20%. It thus appears that the sum rules obtained in model (2) are compatible to the present experimental data.

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EXTRA RESTRICTION ON THE FORWARD SCATTERING AMPLITUDES

Tetsuo Sawada*

International Atomic Energy Agency, International Centre for Theoretical Physics, Trieste, Italy
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One of the most remarkable properties of the second-sheet function of the scattering amplitude, defined by the analytical continuation through the elastic cut, is that we can compute the value of the function itself in a small neighborhood of certain points from the unitarity condition alone without introducing any approximation. This is not the case for the first-sheet function, since, although in a small neighborhood of $s=m^2$ the pole term $g^2/(m^2-s)$ dominates, there still remains finite background contribution $A(s,z)-g^2/(m^2-s)$ and we have no way to compute it exactly. In this note, we restrict ourselves to the case of pion-nucleon scattering and neglect the spin of the nucleon for the reason of simplicity. Our claim is that in a small neighborhood of the point

$$s = s_+ \equiv m^2 + 2\mu^2, \quad (1)$$

where m and μ are the masses of the nucleon and pion, respectively, the second-sheet function of the forward scattering amplitude has the form

$$A^{\mathrm{II}}(s,1) = \frac{g^2}{m^2-u(z=1)} + \frac{g^2C_1}{s_+-s} + g^2C_0' + O\left(\left|\ln\frac{s_+-s}{m^2}\right|^{-2}\right), \quad (2)$$

where C_1 and C_0 can be expressed in terms of the μ, m, and the coupling constant $g^2/4\pi$ explicitly. [See Eqs. (29), (30), (31), and (36).] $s=s_+$ is the point where the existence of the logarithmic singularity has been known from perturbation theory.[1]

Suppose that the total cross section of π-N scattering $\sigma_{\rm tot}(s)$, the coupling constant $g^2/4\pi$, and the subtraction constant a_0 (if necessary) are given accurately; then from the dispersion relation of the forward scattering amplitude,[2] an analytic function $A(s)$ is defined in the cut s plane. It is convenient to introduce q (the magnitude of the momentum in the c.m. system) as a new variable which is related to s by

$$4q^2=(1/s)\{s-(m+\mu)^2\}\{s-(m-\mu)^2\},$$

and the point $s=s_+$ of the second sheet is

$$q=q_+\equiv -i\mu\left[\frac{1-(\mu^2/4m^2)}{1+(2\mu^2/m^2)}\right]^{1/2} \tag{3}$$

in the q plane. The function $A[s(q)]$ is at first defined in the upper half plane of q, but, however, we can make an analytical continuation to the lower half plane of q. We shall denote by $\tilde{A}[s(q)]$ the over-all analytical function which coincides with $A[s(q)]$ in the upper half plane. From Eq. (2), as in polology,[3] we obtain

$$[s_+-s(q)]\left\{\tilde{A}[s(q)]-\frac{g^2}{m^2-u(z=1)}\right\}\Bigg|_{q=q_+}=g^2C_1 \tag{4}$$

and

$$\left\{\tilde{A}[s(q)]-\frac{g^2}{m^2-u(z=1)}-\frac{g^2C_1}{s_+-s(q)}\right\}\Bigg|_{q=q_+}=g^2C_0'. \tag{5}$$

It is evident that Eq. (4) gives rise to a restriction on $g^2/4\pi$ and $\sigma_{\rm tot}(s)$, and Eq. (5) gives a relation among $g^2/4\pi$, a_0, and $\sigma_{\rm tot}(s)$ [in the case of no subtraction this also gives a relation between $g^2/4\pi$ and $\sigma_{\rm tot}(s)$].

We shall give a brief proof of Eq. (2). Assuming the spectral representation for $A(s,t)$ to be[4]

$$A(s,t)=\frac{g^2}{m^2-u}+\frac{g^2}{m^2-s}+b_0(s)+\frac{t-t_0}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{A_t(s,t')}{(t'-t_0)(t'-t)}+\frac{u-u_0}{\pi}\int_{(m+\mu)^2}^{\infty}du'\frac{A_u(s,u')}{(u'-u_0)(u'-u)}, \tag{6}$$

the partial wave amplitudes $a_l(s)$ are given by

$$A_l(s)=-\frac{g^2}{2q^2}Q_l\left(1+\frac{m^2+2\mu^2-s}{2q^2}\right)+\frac{g^2}{m^2-s}\delta_{l,0}+b_0(s)\delta_{l,0}$$
$$+\frac{1}{2q^2}\,\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'A_t(t',s)Q_l\left(1+\frac{t'}{2q^2}\right)-\frac{1}{2q^2}\,\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}du'A_u(u',s)Q_l\left(1+\frac{2m^2+2\mu^2-u'-s}{2q^2}\right). \tag{7}$$

If we fix s in the small neighborhood of $s=s_+$, the argument of the function Q_l in the first integral varies within

$$-\infty<1+\frac{t'}{2q^2}<-1-\frac{18\mu^2}{4m^2-\mu^2}\equiv\sigma_L \tag{8}$$

and the second within

$$\sigma_R\equiv 1+2\frac{(2m+\mu)(m^2+2\mu^2)}{(4m^2\mu+\mu^3)}<1+\frac{2m^2+2\mu^2-u'-s}{2q^2}<\infty, \tag{9}$$

when we carry out the integrations. On the other hand, in the neighborhood of $s=s_+$ the argument of the first Q_l function of Eq. (7) is very close to 1, and so let us introduce a new infinitesimal variable η by

$$1+\frac{m^2+2\mu^2-s}{2q^2}=1+\eta^2. \tag{10}$$

At this point, we list three important properties[5] of the function $Q_l(w)$ which are needed in our proof.

(i) If w is real and $|w|>1$, then

$$|Q_l(w)|<Ce^{-\alpha l}, \tag{11}$$

where

$$\alpha=\cosh^{-1}|w|>0 \tag{12}$$

and C is a finite positive number independent of l and w as long as w lies finitely outside of the region $[-1,+1]$.

(ii) For $|\eta^2|\ll 1$,

$$Q_l(1+\eta^2)$$
$$=K_0(\xi)+(l+\tfrac{1}{2})^{-2}\{[-(1/24)\xi+(13/96)\xi^3]K_1(\xi)+[-\tfrac{1}{12}\xi^2+\tfrac{3}{64}\xi^4]K_0(\xi)\}+O(1/l^4), \tag{13}$$

where

$$\xi=\sqrt{2}\eta(l+\tfrac{1}{2}) \tag{14}$$

and $K_n(\xi)$ is the Bessel function of nth order and the third kind.

(iii) For small $|\eta^2|$,

$$Q_l(1+\eta^2) = -\ln(\eta/\sqrt{2})-\gamma-\psi(l+1)+O(|\eta^2\ln\eta\,|l^2), \quad (15)$$

where

$$\psi(x) \equiv d\ln\Gamma(x)/dx$$

is the poly-gamma function and $\gamma = 0.577\,21\cdots$ is Euler's constant.

Since the l dependence of the first term of the right-hand side of Eq. (7) is quite different from that of the remaining terms, we shall investigate them separately. Let us denote the remaining terms as f_l. Namely, write Eq. (7) as

$$a_l(s) = -(g^2/2q^2)Q_l(1+\eta^2)+f_l. \quad (16)$$

From Eqs. (8) and (9) and from the property of the function Q_l given in Eq. (11),

$$|f_l| < Ce^{-\alpha l} \quad (17)$$

for all non-negative l, where

$$\alpha = \cosh^{-1}|\sigma_L| > 0 \quad (18)$$

and C is a finite positive constant independent of l. Here we assumed that the subtraction function $b_0(s)$ in Eq. (6) is finite at $s=s_+$. Concerning the first term of Eq. (16), it does not damp so fast as f_l when l increases, but it assumes a finite value when l becomes of the order of $1/|\eta|$ as we can see from Eqs. (13) and (14). For extremely large l, namely $l \gg 1/|\eta|$, $Q_l(1+\eta^2)$ also damps. From Eq. (15), $Q_l(1+\eta^2)$ is of the order of $|\ln\eta|$ for $0 \le l \le L$, where L is a positive integer which satisfies $1 \ll L \ll |\eta|^{-1}|\ln\eta|^{-1/2}$. For example we can choose L as

$$L = [|\eta|^{-1/2}]. \quad (19)$$

From the unitarity condition of the partial-wave amplitudes,

$$a_l(s) - a_l^\dagger(s) = iqs^{-1/2}a_l(s)a_l^\dagger(s), \quad (20)$$

for

$$(m+\mu)^2 \le s < (m+2\mu)^2,$$

we can construct the second-sheet function

$$A^{\rm II}(s,z) = \sum_{l=0}^{\infty}(2l+1)\frac{a_l(s)}{1+iqs^{-1/2}a_l(s)}P_l(z). \quad (21)$$

We restrict z to the region $-1 \le z \le 1$, where the series of Eq. (21) converges absolutely. Putting Eq. (16) into Eq. (21) let us examine the contribution of f_l to $A^{\rm II}(s,z)$. Dividing the summation range into $0 \le l \le L$ and $L+1 \le l < \infty$ by using L given in Eq. (19),

$$\sum_{l=L+1}^{\infty}(2l+1)\frac{a_l(s)}{1+iqs^{-1/2}a_l(s)}P_l(z) = -\frac{g^2}{2q^2}\sum_{l=L+1}^{\infty}(2l+1)\frac{Q_l(1+\eta^2)}{1+a^{-1}Q_l(1+\eta^2)}P_l(z)+O(e^{-\alpha L}), \quad (22)$$

since f_l is bounded by $Ce^{-\alpha l}$. In Eq. (22) $1/a$ is defined by

$$\frac{1}{a} = -i\frac{g^2}{2qs^{1/2}}. \quad (23)$$

On the other hand,

$$\sum_{l=0}^{L}(2l+1)\frac{a_l(s)}{1+iqs^{-1/2}a_l(s)}P_l(z) = -\frac{g^2}{2q^2}\sum_{l=0}^{L}\frac{(2l+1)Q_l(1+\eta^2)}{1+a^{-1}Q_l(1+\eta^2)}P_l(z)+iqs^{-1/2}\sum_{l=0}^{L}\frac{(2l+1)f_lP_l(z)}{\{1+iqs^{-1/2}a_l(s)\}\{1+a^{-1}Q_l(1+\eta^2)\}}; \quad (24)$$

since $Q_l(1+\eta^2)$ and $a_l(s)$ are of the order of $|\ln\eta|$ in this range of l, and $\sum(2l+1)f_lP_l(z)$ converges, the second term on the right-hand side of Eq. (24) is of the order of $|\ln\eta|^{-2}$. Combining Eqs. (22) and (24),

$$A^{\rm II}(s,z) = -\frac{g^2}{2q^2}\sum_{l=0}^{\infty}(2l+1)\frac{Q_l(1+\eta^2)P_l(z)}{1+a^{-1}Q_l(1+\eta^2)}+O(|\ln\eta|^{-2}). \quad (25)$$

Thus, f_l cannot give rise to a finite contribution to the second-sheet function in the small neighborhood of $s=s_+$; this situation should be compared with the case of the first-sheet function

$$A(s,z)=\sum_{l=0}^{\infty}(2l+1)\left\{-\frac{g^2}{2q^2}Q_l(1+\eta^2)+f_l\right\}P_l(z), \qquad (26)$$

where f_l gives a finite contribution to $A(s,z)$ at $s=s_+$.

Separating the u-pole term, Eq. (25) becomes

$$A^{\rm II}(s,z)=\frac{g^2}{m^2-u}+\frac{g^2}{2q^2}\frac{1}{a}\sum_{l=0}^{\infty}(2l+1)\frac{[Q_l(1+\eta^2)]^2}{1+a^{-1}Q_l(1+\eta^2)}P_l(z)+O(|\ln\eta|^{-2}). \qquad (27)$$

For $|\eta|\ll 1$ the l summation of Eq. (27) can be converted to an integral by using ξ defined in Eq. (14) as an integral variable. In particular, for $z=1$, the result is

$$\frac{1}{a}\sum_{l=0}^{\infty}(2l+1)\frac{[Q_l(1+\eta^2)]^2}{1+a^{-1}Q_l(1+\eta^2)}=\frac{1}{\eta^2}C_1+C_0+O\left(\frac{1}{|\ln\eta|^2}\right), \qquad (28)$$

where

$$C_1=\frac{1}{a}\int_0^{\infty}d\xi\frac{\xi[K_0(\xi)]^2}{1+a^{-1}K_0(\xi)}, \qquad (29)$$

and

$$C_0=\frac{1}{a}\left[\int_0^{\infty}d\xi\frac{1}{\xi}\left\{G(a,\xi)-\frac{1}{24}F(a,\xi)\right\}+\sum_{l=0}^{\infty}H_l\right]. \qquad (30)$$

In our case,

$$1/a=(g^2/2\mu m)(1-\mu^2/4m^2)^{-1/2}. \qquad (31)$$

In Eq. (30),

$$F(a,\xi)=2a\xi^2\left[K_0(\xi)-\frac{1}{\xi}K_1(\xi)\right]\left\{1-\frac{1}{[1+a^{-1}K_0(\xi)]^2}\right\}+\frac{4\xi^2[K_1(\xi)]^2}{[1+a^{-1}K_0(\xi)]^3}, \qquad (32)$$

$$G(a,\xi)=\frac{2K_0(\xi)}{[1+a^{-1}K_0(\xi)]^2}\left\{\left(-\frac{1}{24}\xi+\frac{13}{96}\xi^3\right)K_1(\xi)+\left(-\frac{1}{12}\xi^2+\frac{3}{64}\xi^4\right)K_0(\xi)\right\}, \qquad (33)$$

and

$$H_l=-2a[(l+\tfrac{1}{2})\psi(l+1)-\{\tfrac{1}{2}l^2\ln(1+1/l)-\tfrac{1}{2}(l+\tfrac{1}{2})+(l+\tfrac{1}{2})\ln(l+1)\}]; \qquad (34)$$

in particular,

$$H_0=-a\psi(1)-\tfrac{1}{2}a=a(\gamma-\tfrac{1}{2}), \qquad (35)$$

where $K_n(\xi)$ and $\psi(x)$ are the same functions as those appearing in Eqs. (13) and (15), respectively. The convergence of the integrals and summation in Eqs. (29) and (30) can easily be checked. From Eqs. (28) and (27) we obtain the final result: Eq. (2), where

$$C_0'=-\frac{1}{2\mu^2}\frac{1+2\mu^2/m^2}{1-\mu^2/4m^2}C_0. \qquad (36)$$

It is worthwhile to point out here that the conversion of the summation to the integral given in Eq. (28) does not hold at the point of poles $\eta=\eta(l,a)$ which satisfies

$$1+(1/a)Q_l[1+\eta^2(l,a)]=0 \qquad (37)$$

and its extremely small neighborhood [circle of radius $|\eta(l,a)|^{4/3}$, for example] in the η plane.

In this note, we have developed a new method which may be used not only to obtain the relation between the total cross section and the coupling constant, but also to compute the subtraction constant from the total cross section without introducing any approximation. As is well known, mathematically, the point $s=s_+$ is an essential singularity on the second sheet, since it is an accumulation point of a series of poles.[6] However, this does not raise any difficulty in our method, as long as the total cross section is known accurately. The technical problem as to how to apply this method when the experimental data of the total cross section are given with some ambiguity will be discussed in the next paper, and the numerical result for the case of pion-nucleon scattering will also be given.

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*Permanent address: Department of Physics, Tokyo University of Education, Tokyo, Japan.

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