is zero unless $j_1(p_1)|0\rangle$ is a physical state, i.e., $p_1^2 \geq M_1^2 > m_2^2$, and so this term cannot contribute to the right-hand side. The second term,

$$\langle p_2|j_1(p_1)j^0(0,q^0)|0\rangle,$$

must therefore contribute to $q^0=0$; thus there exist some states with the quantum numbers of $\varphi_1^*\varphi_2$ and arbitrarily small energy. Q.E.D.

We can immediately apply the theorem to the proton-neutron mass difference. Thus $M_n > M_p$, and considering simply $\langle p|[Q,\psi_n]|0\rangle$ we see that there must exist charge-one particles with arbitrarily small mass, which have never been seen. Similarly, one may not explain the breaking of the SU(3) mass multiplets by spontaneous breakdown.

*The research reported in this document has been sponsored in part by the Air Force Office of Scientific Research through the European Office of Aerospace Research, U. S. Air Force.

[1]J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).

[2]G. S. Guralnik, C. R. Hagen, and T. W. Kibble, Phys. Rev. Letters 13, 585 (1964).

[3]R. F. Streater, to be published.

[4]S. Glashow, Phys. Rev. 130, 2132 (1963).

[5]For a good review of the situation to date, see G. Guralnik, Lectures at the Conference on Unified Field Theory, Munich, July 1965 (to be published).

PSEUDOSPIN MODEL OF LIQUID He^4

P. R. Zilsel

Palmer Physical Laboratory, Princeton University, Princeton, New Jersey,*
and Department of Physics, Stanford University, Stanford, California†

(Received 16 August 1965)

A simple pseudospin model for a gas of hard-core bosons with attractive interactions was introduced by Matsubara and Matsuda,[1] and its zero-temperature properties were discussed by Witlock and Zilsel.[2]

The purpose of this Letter is to point out that, with a slight modification, the model is applicable to liquid He^4 and appears to account, at least semiquantitatively, for the liquid-vapor equilibrium as well as for the λ transition, the off-diagonal long-range order[3] and superfluidity of the low-temperature liquid phase, and the nature of the elementary excitation spectrum.

The model is a quantum generalization of the classical lattice-gas (Ising) model.[4] Alternatively, it may be viewed as a cell approximation to Siegert's[5] formalism for hard-core boson fields: The volume Ω of the system is divided into M cubical cells of size d^3 where d is of the order of, but somewhat larger than, the hard-core radius a. (We shall take $d=2.85$ Å, which makes $a/d \simeq \frac{5}{6}$.) Field amplitudes ψ_j, ψ_j^+ $(j=1,\cdots,M)$, are associated with each cell (rather than each point in space), and the hard-core constraint is represented approximately by $\psi_j^2=0$, or $n_j^2=n_j$, where $n_j=\psi_j^+\psi_j$ is the particle number operator for the jth cell. This means that the states for any one cell are those of a Fermi oscillator. However, the amplitudes for different cells commute, the underlying field being Bose, so that the commutation relations are

$$[\psi_i,\psi_j]_-=0,\quad [\psi_i,\psi_j^+]_-=(1-2n_j)\delta_{ij}. \tag{1}$$

The algebra defined by (1) is that of the generators of an SU(2), the symmetry being between particles and vacancies in each cell. Using the notation of reference 2, we define pseudospin components

$$\sigma_j^{(1)}=\psi_j+\psi_j^+,\quad \sigma_j^{(2)}=i(\psi_j^+-\psi_j),\quad \sigma_j^{(3)}=1-2n_j, \tag{2}$$

which have all the properties of Pauli operators.

In the original version of the model,[1,2] the continuum kinetic-energy operator $(\hbar^2/2m)\times\int\nabla\psi^+\cdot\nabla\psi d^3r$ is replaced by its finite difference approximation

$$\begin{aligned}K&=(\hbar^2/2md^2)\textstyle\sum_{\langle ij\rangle}(\psi_i^+-\psi_j^+)(\psi_i-\psi_j)\\&=(\hbar^2/md^2)\textstyle\sum_{\langle ij\rangle}[\frac{1}{4}(1-\vec{\sigma}_i\cdot\vec{\sigma}_j)+n_in_j],\end{aligned} \tag{3}$$

and the attractive part of the potential energy by a "square well" attraction

$$V=-v\textstyle\sum_{\langle ij\rangle}n_in_j, \tag{4}$$

where $\langle ij\rangle$ stands for nearest-neighbor pairs in the cubic lattice space. (The hard-core repulsion is, of course, represented in the com-

mutation relations.) Using energy units $\hbar^2/md^2 = 1.49°K$ ($d = 2.85$ Å), the Hamiltonian is, then,

$$H = \sum_{\langle ij \rangle} [\tfrac{1}{4}(1 - \vec{\sigma}_i \cdot \vec{\sigma}_j) + A n_i n_j], \qquad (5)$$

with

$$A = 1 - v(md^2/\hbar^2).$$

The limit $A \to -\infty$ (i.e., $\hbar \to 0$) is the classical lattice gas.[4] The case $A = 0$ ("isotropic Heisenberg ferromagnet"), as discussed in reference 2, corresponds to a system with zero scattering length; the "square well" of depth $\hbar^2/md^2$ just cancels the phase shift due to the hard core for zero wave number. In this case of exact SU(2) symmetry, H commutes with the total pseudospin $\vec{S} = \frac{1}{2}\sum_j \vec{\sigma}_j$. The ground state for any particle number N is the state of maximal $S(=M/2)$ (M = number of cells $= \Omega/d^3$) with three component $S^{(3)} = (M/2) - N$. The ground-state energy $E_0 = 0$, independent of N. If we fix N only in the mean (grand ensemble; $\langle S^{(3)} \rangle = \frac{1}{2}M - \langle N \rangle$), we have a degenerate many-body ground state with maximal pseudospin alignment, $S = \frac{1}{2}M$, at an angle θ with the three direction such that $\langle S^{(3)} \rangle = \frac{1}{2}M \cos\theta$. The degeneracy is due to the arbitrary direction of $S_\perp$ in the 1-2 plane [broken SU(2) symmetry], just as in Anderson's pseudospin formulation[6] of BCS. There is off-diagonal long-range order due to the macroscopic value of the transverse pseudospin component $S_\perp$. As shown in reference 2, this corresponds to incomplete Bose condensation.

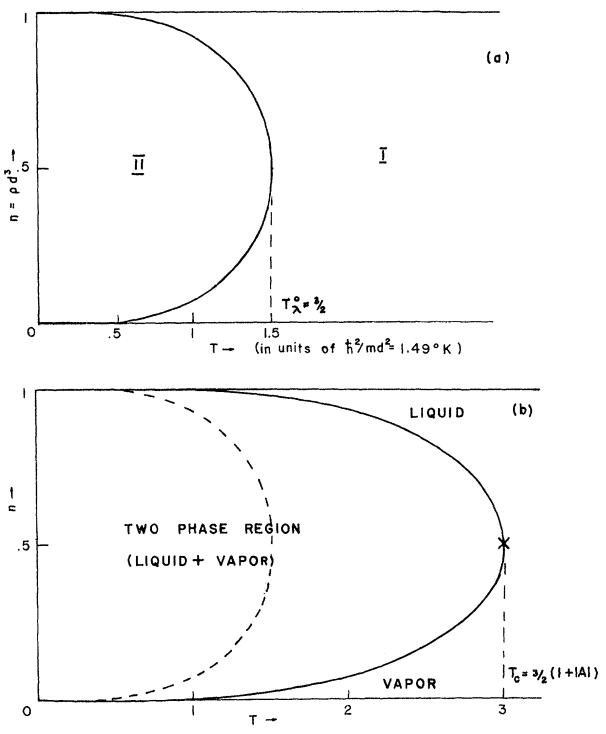

FIG. 1. ρ-T diagram for the simple pseudospin model (references 1 and 2) in "molecular field" approximation. (a) For $A \geq 0$ (gaseous ground state). Region I corresponds to the "normal" gas, region II to "Bose condensation." (b) For $A < 0$ ("liquid" ground state). The full line represents the liquid and vapor densities in equilibrium. The region inside this curve corresponds to the two-phase region. The λ transition (broken line) is not real in this case since it lies entirely within the unstable two-phase region. The two-phase curve shown in the figure is for $A = -1$.

The case of (small) positive A is qualitatively similar. This case corresponds to positive scattering length and purely positive eigenvalue spectrum of the Hamiltonian (5) (i.e., a gas). H no longer commutes with $\vec{S}$ (but still with $S^{(3)}$, i.e., with N; axially symmetric Heisenberg ferromagnet). The ground state is still homogeneous, though the ground-state energy per particle E_0/N is now a positive increasing function of $n = N/M = \rho d^3$. For finite $n < 1$, there is at low temperatures still a tendency to pseudospin alignment, the mean magnetization per cell, $\sigma = 2\langle S \rangle/M$, being larger than $|\langle \sigma^{(3)} \rangle|$, which is constrained to

$$\sigma^{(3)} = 1 - 2n. \qquad (6)$$

(The symbols $\langle\ \rangle$ now denote thermal expectation values.) For simplicity we discuss the molecular field (Hartree-Fock) approximation. In this approximation the Helmholtz free energy per cell is given by

$$f = (\tfrac{3}{4})(1 - \sigma^2) + 3An^2 - T \times \{\ln 2 - \tfrac{1}{2}[(1+\sigma)\ln(1+\sigma) + (1-\sigma)\ln(1-\sigma)]\}, \qquad (7)$$

which, for fixed n, is minimized by the Brillouin function $\sigma = \tanh(3\sigma/2T)$. Since σ must be $\geq |\sigma^{(3)}|$ as given by (6), the phase transition for a given density occurs at the temperature $T_\lambda(\rho)$ at which

$$|\sigma^{(3)}| = |1 - 2n| = \tanh[(3|1-2n|)/2T_\lambda]. \qquad (8)$$

The transition temperature is highest for $n = \frac{1}{2}$: $T_\lambda^0 = T_\lambda(n = \frac{1}{2}) = \frac{3}{2}(=2.24°K)$. The transition (Curie point) is second order in this approximation, but an exact solution would presumably show a logarithmic singularity in the specific heat. The phase diagram is shown in Fig. 1(a). It is symmetric about the line $n = \frac{1}{2}$. For $n < \frac{1}{2}$ the transition may be thought of, roughly, as a Bose condensation of particles, for

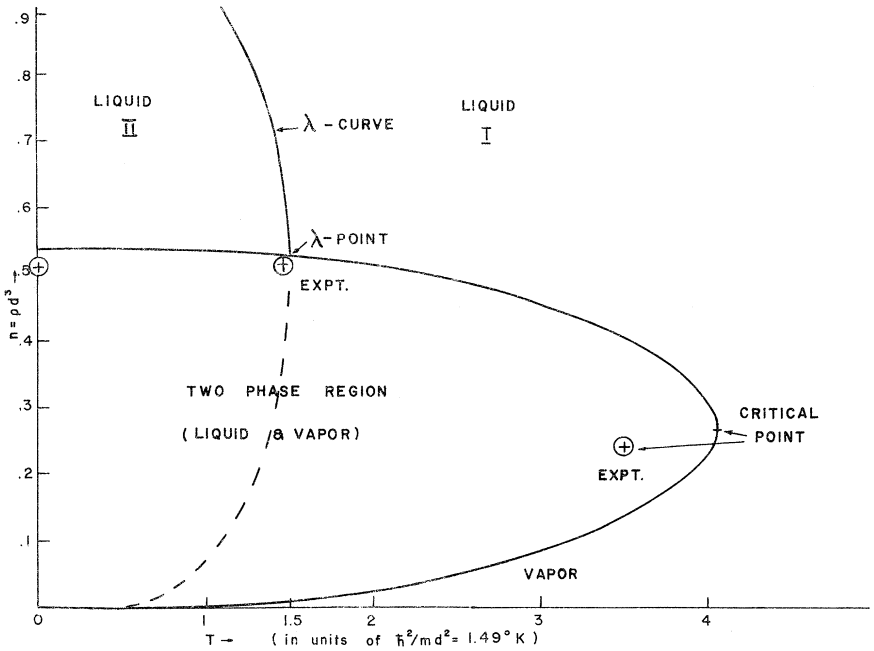

FIG. 2. ρ-T diagram with zero-point energy correction [Eq. (7′)] in "molecular field" approximation, for $A=-3$, $a/d=\frac{5}{6}$. The solid portion of the λ curve lies within the liquid phase and is real in this case. The broken portion, in the unstable two-phase region, is not real.

$n>\frac{1}{2}$ as a Bose condensation of holes. Note, however, that in contrast to the ideal Bose gas, the present system has a phase transition also in the two-dimensional case[7] (thin film), the λ temperature being reduced (in the molecular field approximation) by a factor $\frac{2}{3}$.

For negative A (negative scattering length; bound many-body ground state), the situation is quite different. Except for negligible surface terms the ground-state energy per particle is $-3|A|n$, so that the system tends to maximum density ($n=1$). For $N/M<1$ (i.e., $|S^{(3)}|<\frac{1}{2}M$), the system separates into two phases (domains): a condensed high-density phase with, at $T=0$, $n_l=1$ (pseudospins "down"), and a low-density gas phase with, at $T=0$, $n_g=0$ (pseudospins "up"). For finite T the phase equilibrium is given by

$$\mu_l=\mu_g,\quad \pi_l=\pi_g, \tag{9}$$

where the chemical potential is $\mu=\partial f/\partial n$, and the pressure $\pi(=md^5P/\hbar^2)=n\mu-f$. In the "molecular field" approximation [Eq. (7)] the solution of (9) is again a Brillouin function symmetric about $n=\frac{1}{2}$:

$$n_l(T)=1-n_g(T),$$

$$1-2n_g=\tanh\{(\tfrac{3}{2})(1+|A|)(1-2n_g)/T\}. \tag{10}$$

This is shown in Fig. 1(b). The critical point is at $n_g=n_C=\frac{1}{2}$, $T_C=(\frac{3}{2})(1+|A|)=(1+|A|)T_\lambda{}^0$. The essential feature is that for any $A<0$ the curve (10) is everywhere outside the λ curve (8), so that no λ transition occurs (the "superfluid" region lies in the unstable two-phase portion of the ρ-T diagram): In the gas phase there are not enough particles, in the condensed phase not enough holes, for Bose condensation to occur.

Modified Hamiltonian.—Any modification of the Hamiltonian (5) which makes the ground-state density of the liquid phase, n_l, less than 1 and produces an intersection of the λ curve with the liquid-gas coexistence curve in the ρ-T plane will result in a phase diagram showing both the λ transition and the liquid-gas transition. This may be accomplished in several ways. The simplest and, I believe, physically correct way is to note that the finite difference approximation (3) for the kinetic energy smoothes out the short-wavelength behavior of the wave field and thus does not give the high-density pole in the zero-point energy[8]: $K_0/N\to\pi^2\hbar^2/2m(r-a)^2$, as $r-a\to 0$ ($r=$nearest-neighbor distance). In the molecular field approximation this may be written as $K_0/N\to(\pi^2\hbar^2/2m)\rho^{2/3}(1$

Table I. Summary of numerical results for pseudospin model with zero-point energy correction, in "molecular field" approximation. Parameters (defined in the text): $A=-3$, $d=2.85$ Å, $a/d=\frac{5}{6}$, $\hbar^2/md^2=2.06\times10^{-16}$ erg = 1.49°K).

	Calculated	Experimental
λ temperature	$(\frac{3}{2})\hbar^2/md^2=2.24$°K	2.19°K
Critical temperature	$4.1\hbar^2/md^2=6.1$°K	5.2°K
Pressure at λ point	0.12 atm	0.05 atm
Critical pressure	3.8 atm	2.3 atm
Liquid ground-state energy per atom	$-4.0\hbar^2/md^2=-8.2\times10^{-16}$ erg	-10×10^{-16} erg
Liquid ground-state density	2.3×10^{22} cm^{-3}	2.2×10^{22} cm^{-3}
Speed of sound (from compressibility at $T=0$)	270 m/sec	240 m/sec

$-a\rho^{1/3})^{-2}=(\pi^2\hbar^2/2md^2)n^{2/3}(1-an^{1/3}/d)^{-2}$. We can correct for the presence of this pole by adding to f [Eq. (7)] a term

$$f'=(\tfrac{1}{2})\pi^2n^{5/3}(1-an^{1/3}/d)^{-2}\times y(n). \qquad (7')$$

The factor $y(n)$ is a rapidly increasing function which approaches 1 as $n\to(d/a)^3$ (i.e., $\rho\to a^{-3}$), and approaches 0 as $n\to0$. Its exact form is not important. We choose

$$y(n)=[n(a/d)^3]^3. \qquad (7'')$$

Since f' depends on n only, its presence does not affect the λ curve [Eq. (8)]. The liquid-gas equilibrium [Eq. (9)] must now be computed numerically. The resulting ρ-T diagram is shown in Fig. 2 for $A=-3$ (i.e., $v=8.2\times10^{-16}$ erg), $d=2.85$ Å, and $a/d=\frac{5}{6}$. Some of the numerical results are summarized in Table I. Considering the crudeness of the molecular field approximation, the agreement with experiment is almost too good. Calculations using the temperature-dependent RPA ("spin-wave" approximation) are being done.

Too many people to acknowledge here individually have helped me with discussions. I am particularly grateful to Professor Gerry Brown for the opportunity to work in his group at Princeton and for many stimulating discussions, and to Professor Felix Bloch for his hospitality at Stanford University where this work was finished. The computing aid of Mr. Ben Carter at Stanford is gratefully acknowledged. To Professor Chen Ning Yang, I owe a special debt of gratitude for the benefit of his profound insight and advice.

---

*National Science Foundation Science Faculty Fellow on leave from Western Reserve University, Cleveland, Ohio.

†Work supported in part by the U. S. Navy through Navy Research Contract No. NONR 225(75).

[1]T. Matsubara and H. Matsuda, Progr. Theoret. Phys. (Kyoto) 16, 569 (1956); 17, 19 (1957).

[2]R. Whitlock and P. Zilsel, Phys. Rev. 131, 2409 (1963).

[3]C. N. Yang, Rev. Mod. Phys. 34, 694 (1962).

[4]T. D. Lee and C. N. Yang, Phys. Rev. 87, 410 (1952); C. N. Yang and C. P. Yang, Phys. Rev. Letters 13, 303 (1964).

[5]A. J. F. Siegert, Phys. Rev. 116, 1507 (1959).

[6]P. W. Anderson, Phys. Rev. 112, 1900 (1958).

[7]I am indebted to V. J. Emery for pointing this out.

[8]Cf. F. London, Superfluids (John Wiley & Sons, Inc., New York, 1954), Vol. II, p.30. I am grateful to Victor Emery for reminding me that this pole is important at the liquid density.