ing the region of minimum B, it would be likewise forced around the toroid.

At the collision point the approaching electric octupoles caused a circumferential electric field between them. This field was observed and the $E\times B$ drift was to the wall. For a brief time, $\sim 2\times 10^{-6}$ sec, some plasma was observed by the ion-energy analyzer to hit the wall, and the density probes showed plasma moving out toward the wall. However, all plasma which entered the multipole did not escape since the remainder which was confined after the first 50-μsec filling period seemed to be trapped with more than 10% capture efficiency. The ion analyzer also detected protons thrown outward at other azimuths during the filling time —usually in short bursts spaced in time but coming later than the collision.[6]

Where the collision occurred and at some other points a transient inversion of density gradient, opposite to the direction of the theoretically stable gradient, sometimes occurred. When this happened, a train of electric-field oscillations of the order of 1 Mc/sec in frequency was generated during the density readjustment.

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GENERAL CRITERION FOR ELECTROSTATIC PLASMA INSTABILITIES WITH UNIFORM MAGNETIC FIELD*

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Despite extensive research[1] in the theory of microinstabilities in plasmas, it has been possible until now to give a necessary and sufficient criterion for instability[2,3] only in the case of electrostatic plasma oscillations in a uniform plasma without magnetic field. The criterion, first derived in rigorous form by Penrose,[2] states that exponentially-growing electrostatic plasma oscillations can exist if, and only if, there is a speed u_0 such that[4]

$$\left.\frac{dF}{du}\right|_{u=u_0} \equiv F'(u_0)=0, \text{ with } F'(u_0-\epsilon)<0, \quad F'(u_0+\epsilon)>0, \tag{1}$$

and

$$\oint_{-\infty}^{\infty}\frac{duF'(u)}{u-u_0}>0, \tag{2}$$

where

$$F(u)=\sum_j \omega_{pj}^{2}\int d^3\vec{v}'\,\delta[(\vec{v}'\cdot\vec{k}/|\vec{k}|)-u]f_{0j}(\vec{v}') \tag{3}$$

and the sum over j runs over all the charged species present. Thus the microscopic stability of a uniform plasma without magnetic field is completely determined (except for possible nonexponential growth, as discussed by Penrose) by the properties of the "projection" (3) of the distribution functions $f_{0j}(\vec{v})$ of the various species of the undisturbed plasma. This condition is conveniently derived in terms of a Nyquist-diagram technique in complex ω/k space. In the above formulas $\vec{k}$ is the propagation vector of the disturbance.

The purpose of this note is to show that a similar general criterion can be given for electrostatic (longitudinal) plasma oscillations in the presence of a uniform magnetic field. Having limited ourselves to electrostatic disturbances (i.e., in a restricted sense to the nonrelativistic limit of the general dispersion relation[1,5,6]), we begin with the "dispersion relation" essentially in the form given by Harris[6]:

$$1=\frac{1}{k^2}\int_{-\infty}^{\infty}dv_z\int_0^{\infty}2\pi v_\perp dv_\perp\sum_j\sum_{n=-\infty}\frac{\omega_{pj}^2J_n^2(k_\perp v_\perp/kw_{cj})[(k_z/k)(\partial f_{0j}/\partial v_z)+(nw_{cj}/v_\perp)(\partial f_{0j}/\partial v_\perp)]}{(k_zv_z/k)+nw_{cj}-w},\tag{4}$$

where

$$\mathrm{Im}(w)>0,$$

and we have used the usual cylindrical coordinate system with $\vec{B}$ in the z direction. In this expression, $\omega_{pj}^2=\epsilon_0^{-1}n_0e^2/m_j$, $w_{cj}=|z_je\vec{B}|/m_jk$, $k=+(k_z^2+k_\perp^2)^{1/2}$, $w=\omega/k$, where ω is the frequency of the disturbance with time dependence $e^{-i\omega t}$ and each $f_{0j}=f_{0j}(v_z,v_\perp)$. $J_n(x)$ is a Bessel function of the first kind with real argument.

Noting that v_z appears in (4) only as an integration variable, we may conveniently make the transformation, for each n and j (provided $k_z\neq0$, see below),

$$(k_zv_z/k)+nw_{cj}=u,\tag{5}$$

$$f_{0j}(v_z,v_\perp)=f_{0j}[(k/k_z)(u-nw_{cj}),v_\perp],\tag{6}$$

and immediately rewrite (4) in the form

$$1=\frac{1}{k^2}\int_{-\infty}^{\infty}\frac{duP(u)}{u-w};\ \mathrm{Im}(w)>0,\tag{7}$$

where

$$P(u,k_z,k_\perp)=\int_0^{\infty}2\pi v_\perp dv_\perp\sum_j\sum_{n=-\infty}\omega_{pj}^2J_n^2\left(\frac{k_\perp v_\perp}{kw_{cj}}\right)$$

$$\times\left\{\frac{\partial f_0}{\partial v_z}(v_z,v_\perp)\Big|_{v_z=(k/k_z)(u-nw_{cj})}+\frac{nw_{cj}}{v_\perp}\frac{k}{k_z}\frac{\partial f_{0j}}{\partial v_\perp}\left[\frac{k}{k_z}(u-nw_{cj}),v_\perp\right]\right\}\mathrm{sgn}(k_z).\tag{8}$$

We note that ku is a Doppler-shifted cyclotron frequency or one of its harmonics, and $P(u)$ is a generalized "projection" of the derivatives of the f_{0j} which specifically brings in the successively higher weighted moments of the f_{0j} in their $v_\perp$ dependence. In general, positive and negative k_z must be treated separately, i.e., the properties of $P(u,k_z,k_\perp)$ must be investigated for all $k_z\neq0$. In the limit $k_z=0$, however, the second term in $P(u)$ goes over into a sum of δ functions while the first term vanishes, and the original form (4) for $k_z\equiv0$ is recovered. We shall not discuss this limiting case here; it has been extensively studied elsewhere.[1,5] (We also note in passing that for electrostatic waves the case $k_\perp=0$ reduces, as is well known, exactly to the usual form, as if $|\vec{B}|=0$.) When $k_z\neq0$, $P(u)$ is a reasonably smooth function of u, and (7) is exactly of the form studied by Penrose. Moreover, one can show readily that

$$\lim_{u\to\infty}P(u)=0^-,$$

$$\lim_{u\to-\infty}P(u)=0^+.\tag{9}$$

Thus, Eq. (7) has solutions for $\mathrm{Im}(w)>0$, and the plasma can support exponentially growing disturbances, if and only if the Penrose criterion (1) and (2) is satisfied by the generalized projection $P(u)$ in place of $F'(u)$. (Note that with (9) $P(u)$ has the same properties as $F'(u)$

as $|u| \to \infty$, and condition (1) becomes simply the requirement that $P(u)$ have more than one zero for $|u| < \infty$.)

We may consider a few examples. If each f_{0j} is Maxwellian,

$$f_{0j} = (\pi^{3/2}\alpha_j^{\ 3})^{-1} \exp(-v^2/\alpha_j^{\ 2}), \tag{10}$$

it follows immediately that $P(u)$ has only one zero (at $u = 0$) and the plasma is necessarily stable to electrostatic excitation as proved by Newcomb and Bernstein.[5]

An especially interesting example is provided by the anisotropic distribution studied by Harris[6] and many others[7]:

$$f_{0j} = (\pi^{3/2}\alpha_{\perp j}^{\ 2}\alpha_{zj})^{-1} \exp(-v_\perp^{\ 2}/\alpha_{\perp j}^{\ 2}) \times \exp(-v_z^{\ 2}/\alpha_{zj}^{\ 2}). \tag{11}$$

For simplicity, we let the ion mass become very large and consider only the electrons. Then, after some simple algebra and use of well-known formulas for certain integrals over Bessel functions,[8] we find

$$P(u) = -2\exp\left(-\frac{k_\perp^{\ 2}R_c^{\ 2}}{2}\right)\sum_{n=-\infty}^{\infty} I_n\left(\frac{k_\perp^{\ 2}R_c^{\ 2}}{2}\right) \times \left(\frac{k/k_z}{\pi^{1/2}d_z}\right)\exp\left[-\frac{k^2(u-w_c n)^2}{k_z^{\ 2}\alpha_z^{\ 2}}\right] \times\left[\frac{u}{\alpha_z^{\ 2}} + w_c n\left(\frac{1}{\alpha_\perp^{\ 2}} - \frac{1}{\alpha_z^{\ 2}}\right)\right], \tag{12}$$

where R_c is the electron gyroradius based on the thermal speed $\alpha_\perp$.

Inspection of (12) shows that if $(1/\alpha_\perp^{\ 2} - 1/\alpha_z^{\ 2}) > 0$, i.e., $T_\perp < T_\parallel$, then the contribution of the last term in square brackets to $P(u)$ has the same qualitative form as the first part (only one zero, at $u = 0$), and the plasma is electrostatically stable. On the other hand, if $T_\parallel < T_\perp$, the second part reverses sign, the total $P(u)$ can have more than one zero, and condition (1) can be fulfilled, i.e., the plasma can be unstable. In particular, it is especially appealing that the projection (8) for $T_\parallel \ll T_\perp$ transforms an anisotropic distribution exactly into a form corresponding to the familiar two-stream instability.

As another familiar example, if one includes the ions, inspection of $P(u)$ shows that if $T_e \gg T_i$, one can have ion wave instabilities[2,9] if there is a slight relative displacement of the ion and electron distributions (model of a current-carrying plasma).

The second part of the Penrose criterion provides, in a sense, the "sufficient" condition for instability, if condition (1) is satisfied. Results of the study of condition (2) for various examples will be reported subsequently.

Possession of a general necessary and sufficient criterion for electrostatic instability of a plasma with a uniform magnetic field would seem to be a valuable tool for unifying the extensive literature concerning particular examples of such instability. As implied by the work of Sudan[10] and Noerdlinger,[11] it is possible to utilize similar techniques for classes of transverse waves. Moreover, the existence of such a criterion is intimately connected with the completeness of a set of singular eigenmodes ("stationary solutions") of the linearized Vlasov equation with magnetic field, recently derived. The nature of these eigenmodes, analogous to the Van Kampen–Case modes, as well as various properties of the functional $P(u)$ will be discussed in a paper currently being prepared.

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POLARIZATION DEPENDENCE OF THE PIEZOREFLECTANCE IN Si AND Ge

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Previous high-precision measurements of the piezoreflectance of Si,[1] based on the technique used by Philipp, Dash, and Ehrenreich,[2] have been extended to include the dependence on the polarization of the radiation. Here, results of static and low-frequency measurements are given for the reflection peaks of Ge and Si near 2 and 3.4 eV, respectively, and a microscopic theory for the dependence of the effect on the orientation of the stress axis and on the polarization is described. A more comprehensive discussion is given elsewhere.[3] These data allow one to deduce the symmetry of the transitions responsible for the reflection peaks and the deformation-potential constants. The results are also of interest with regard to more recent measurements[4,5] obtained by an ac technique using piezoelectric transducers.

In both Si and Ge, the stress axes have been chosen to be parallel to the [001] and the [111] directions, and light has been linearly polarized parallel and perpendicular to the stress axes. All curves have been taken at room temperature on crystals immediately after etching in CP4. The data which were reproducible to $\Delta R/R = 3\times10^{-4}$ (error in the relative position of the curves to each other is $\Delta R_{\rm rel} = \pm0.5\,\%$) reveal effects not observed in previous static measurements.[2] Also, dynamical measurements have been carried out on Ge by applying a sinusodial alternating stress at about 100 cps with narrow-band phase-sensitive detection. These give results in good agreement with the static measurements.

The results for the 2-eV peaks of Ge are shown in Fig. 1. In the case of the [001] axis [Fig. 1(a)], there is a polarization-independent energy shift of the reflectivity peaks, whereas a shift with a marked polarization dependence is observed in the [111] case [Fig. 1(b)].

If we assume that transitions with k along $\langle 111\rangle$ axes of the type $\Lambda_4+\Lambda_5\rightarrow\Lambda_6$, $\Lambda_6\rightarrow\Lambda_6$ (double group notation) are primarily responsible[6] for the 2-eV peaks, then [001] stress affects all $\langle 111\rangle$ axes equally and causes no polarization-dependent splitting in energy, as observed. On the other hand, under [111] stress, the eight $\langle 111\rangle$ levels divide into two groups of two and six levels. The energy shifts of the two groups depend on their orientation to the stress axis. The contributions of the two groups of $\langle 111\rangle$ levels to the intensity of the transition are determined by the respective orientations of the polarization vector. The observed dependence of the energy shift on polarization for stress thus supports the above assignment.

From the observed energy shifts one can determine the deformation-potential constants[7-9] $E_{1c}{}^{\Lambda}-E_{1v}{}^{\Lambda}=-5.7$ eV $\pm$ 5%, $E_{2c}{}^{\Lambda}=+5.1$ eV $\pm$ 20%, where c and v refer to the conduction and valence bands, respectively. The value for $E_{1c}{}^{\Lambda}-E_{1v}{}^{\Lambda}$ is the same as that deduced from hydrostatic measurements by Paul.[10]

In addition to the energy shifts of the reflectivity peaks, there is a strong strain-induced change in the magnitude of the peaks [Fig. 1(a)]. To explain this effect, we consider the imaginary part of the dielectric constant, ϵ_2, which shows essentially the same structure as the reflectivity in the region of strong absorption. Both the joint density of states[11] and the transition matrix elements contribute to ϵ_2. Each of these factors may change by the application of uniaxial stress. In the case of [001] stress, the reflectivity of the 2.1-eV peak increases, and the reflectivity of the 2.3-eV peak decreases for light polarized parallel to the compression axis, whereas the converse occurs for the perpendicular polarization (Fig. 1). The change in the joint density of states caused by [001]