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LIE GROUP OF THE STRONG-COUPLING THEORY*

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The purpose of this paper is three fold: (i) to show the Lie group of Goebel's theory[1] of strong coupling in static models; from this group the bands of isobar states are obtained as irreducible representations [they are of infinite dimension because the Lie group is noncompact]; (ii) to give a solution of the problem for two simple cases [symmetric scalar and symmetric pseudoscalar meson theory] by the help of the method of group contraction; and (iii) to indicate the possibility of higher symmetries. We will briefly sketch here the essential part of the strong-coupling theory, which serves as the physical derivation of the algebra. Consider the scattering of a scalar meson by a static isobar: "α" + "i" → "β" + "j", where α and β indicate the initial and final mesons, respectively, while i and j are isobar states. The Chew-Low[2] equation for this process is

$$T_{\beta\alpha}{}^{ji}(\omega) = -\lambda^2 \sum_k \left[\frac{(A_\beta)^{jk}(A_\alpha)^{ki}}{M_k - M_i - \omega} + \frac{(A_\alpha)^{jk}(A_\beta)^{ki}}{M_k - M_j + \omega}\right] - \sum_{\vec{p},k}\left\{\frac{[T_{\gamma\beta}{}^{kj}(\omega_p)]^* T_{\gamma\alpha}{}^{ki}(\omega_p)}{M_k + \omega_{\vec{p}} - M_i - \omega} + \frac{[T_{\gamma\alpha}{}^{kj}(\omega_p)]^* T_{\gamma\beta}{}^{ki}(\omega_p)}{M_k + \omega_{\vec{p}} - M_j + \omega}\right\}$$

$$+ (\text{two or more meson intermediate states}), \qquad (1)$$

where M_i is the energy of the ith isobar and $\lambda(A_\alpha)^{ij}$ is the matrix element of the source of the meson α between the ith and jth isobar. The parameter λ represents the strength of the coupling, the A_α being kept finite as the strong-coupling limit ($\lambda^2 \to \infty$) is taken. In the strong-coupling limit all isobars are degenerate, $M_i \to M + O(1/\lambda^2)$, and so we write $M_i = M + \Delta_i/\lambda^2$, where Δ_i is kept finite. The scattering amplitude is finite in the physical region because of its unitarity property, so the pole terms [first two terms in (1)] must be finite in the strong-coupling limit. Expanding the pole terms in powers of $1/\lambda^2$, one obtains

$$T_{\beta\alpha}{}^{ji}(\omega)_{\text{pole}} \sim -\frac{\lambda^2}{\omega}([A_\beta, A_\alpha])^{ji} - \frac{1}{\omega^2}([A_\beta,[\Delta, A_\alpha]])^{ji} + O\left(\frac{1}{\lambda^2}\right), \qquad (2)$$

where matrix notation is used for A's and Δ is a diagonal matrix with Δ_i as the ith diagonal element. To keep T_{pole} finite, in the strong-coupling limit one must have

$$[A_\beta, A_\alpha] = 0. \qquad (3)$$

This equation is equivalent to the coupling condition of reference 1; it is a "bootstrap" condition.

We suppose now that there is an invariance group K (isotopic spin, for example), such that the isobar states form a basis of a unitary representation of K and the meson sources A_α also form one or more multiplets of the invariance group K. In this representation space, therefore, the A_α constitute a tensor operator of the group K. Let X_i be the infinitesimal generators of the group K. Then

$$[X_i, X_j] = C_{ij}{}^k X_k,$$

$$[X_i, A_\alpha] = D(i)_\alpha{}^\beta A_\beta,$$

$$[A_\alpha, A_\beta] = 0. \tag{I}$$

The operators X_i and A_α form bases of the Lie algebra G, while the operators A_α form bases of the ideal T of G. Let G and T be the corresponding Lie groups to G and T, respectively, then G is the semidirect product of K and T: $G = K \times T$. Since T is abelian and $G \supset T$, G is noncompact. The basis of an irreducible unitary representation of G is a band of isobar states. The number of isobar states in a band is infinite, because the unitary representations of a noncompact group are of infinite dimension.

Assuming the scattering amplitude has the form

$$T_{\beta\alpha}(\omega) = \frac{\Lambda_{\beta\alpha}}{-\mu - iq} \text{ where } q = (\omega^2 - \mu^2)^{1/2},$$

in the strong-coupling limit, the unitarity relation can be written as

$$\Lambda_{\beta\alpha} = \Sigma_\gamma \Lambda_{\beta\gamma} \Lambda_{\gamma\alpha}, \tag{II.1}$$

and by comparing with Eq. (2) at $\omega \approx 0$ one obtains (setting $\Delta = 2\mu\mathfrak{M}$)

$$\Lambda_{\beta\alpha} = [A_\beta, (\mathfrak{M}, A_\alpha)]. \tag{II.2}$$

(II) is equivalent to the "isobar energy condition" of reference 1.

The representation of G must satisfy the Equations (II) for some $\mathfrak{M}$. It is not known to us whether this is a real restriction on the representations of G. In specific examples that follow, we shall only look for solutions of (I) and (II) in which $\mathfrak{M}$ is proportional to the second-order Casimir operator of K.[3] This restriction on $\mathfrak{M}$ also restricts the representations of G.

(1) Charge-symmetric scalar meson theory.– In this case the relevant invariance group is the isotopic-spin group $K = \mathrm{SU}(2)$. The meson currents $A_\alpha (\alpha = 1, 2, 3)$ are assumed to be a tensor operator of the adjoint representation of K. Therefore, $G = \mathrm{SU}(2) \times T_3$ which is locally isomorphic to the three-dimensional Euclidian group.

In order to find irreducible unitary representations of G, we use the method of group contraction.[4] Consider the Lie algebra of SU(2) $\otimes$SU(2) defined by

$$[L_\alpha^{(1)}, L_\beta^{(1)}] = i\epsilon_{\alpha\beta\gamma} L_\gamma^{(1)},$$

$$[L_\alpha^{(2)}, L_\beta^{(2)}] = i\epsilon_{\alpha\beta\gamma} L_\gamma^{(2)},$$

$$[L_\alpha^{(1)}, L_\beta^{(2)}] = 0. \tag{4}$$

If we set

$$I_\alpha = L_\alpha^{(1)} + L_\alpha^{(2)} \text{ and } A_\alpha = \epsilon(L_\alpha^{(1)} - L_\beta^{(2)}), \tag{5}$$

and let $\epsilon \to 0$ keeping A_α finite, we obtain the Lie algebra G. By this contraction of the Lie algebra we obtain an irreducible unitary representation of G from a representation of the SU(2)$\otimes$SU(2) algebra. The irreducible representation of the SU(2)$\otimes$SU(2) group can be specified by (l_1, l_2), and the Casimir operators of the group $[L^{(1)}]^2$ and $[L^{(2)}]^2$ have the well-known eigenvalues for this representation $[L^{(1)}]^2 = l_1(l_1+1)$, $[L^{(2)}]^2 = l_2(l_2+1)$. This representation can be reduced to the irreducible representations of the isotopic-spin group

$$I^2 = t(t+1) \text{ and } t = |l_1 - l_2|, \cdots, l_1 + l_2. \tag{6}$$

From (5) we obtain

$$4\epsilon^2 [L^{(i)}]^2 = 4\epsilon^2 l_i(l_i+1) = A^2 \pm 2\epsilon \vec{A}\cdot\vec{I} + \epsilon^2 I^2. \tag{7}$$

In order that A^2 is not zero (faithful representation), we must take $l_1 \to \infty$, $l_2 \to \infty$ keeping $t_0 = |l_1 - l_2|$ finite in the limit of $\epsilon \to 0$. In fact, t_0 is given by the invariant operators of the group G, A^2 and $\vec{A}\cdot\vec{I}$, namely $t_0 = \vec{A}\cdot\vec{I}/2(A^2)^{1/2}$. Therefore, the irreducible unitary representation of G is specified by t_0, and it contains an infinite number of irreducible representations of the subgroup SU(2) of G, namely $t_0, t_0+1, \cdots, \infty$.

One finds that

$$\mathfrak{M} = (1/2A^2) I^2$$

satisfies relations (II). Defining the coupling constant by $g^2=\lambda^2A^2$ and using $M_i=M+\Delta_i/\lambda^2$ we obtain

$$M(t)=\frac{t(t+1)}{g^2}\mu+M. \tag{8}$$

(2) Charge-symmetric pseudoscalar meson theory.—In this case, the mesons interact with the isobar in the P state,[5] so that the sources $A_{i\alpha}$ are the tensor components of the adjoint representation of the ordinary-spin group as well as of the isospin group. Therefore the structure of the Lie algebra (I) becomes

$$[I_\alpha,I_\beta]=i\epsilon_{\alpha\beta\gamma}I_\gamma,\ [J_i,J_j]=i\epsilon_{ijk}J_k,$$
$$[I_\alpha,A_{i\beta}]=i\epsilon_{\alpha\beta\gamma}A_{i\gamma},\ [J_i,A_{\alpha j}]=i\epsilon_{ijk}A_{\alpha k},$$
$$[A_{i\alpha},A_{j\beta}]=0, \tag{9}$$

where Greek and Latin indices run over 1, 2, and 3 and refer to isospin and ordinary spin, respectively. The Lie group G is $[SU(2)\otimes SU(2)]\times T_9$. It is now obvious that the irreducible unitary representation of the algebra can be obtained from those of SU(4) by the method of contraction.

The representations of SU(4) can be specified by three integers $(\lambda_1,\lambda_2,\lambda_3)$ which are given by the differences of four integers which specify the representations of U(4) (l_1,l_2,l_3,l_4), where $\lambda_i=l_i-l_4$ $(i=1,2,3)$ and $l_1\geqslant l_2\geqslant l_3\geqslant l_4$. Let C_n be the nth-order Casimir operator of SU(4) $(n=2,3,4)$, whose eigenvalue is an nth order polynomial of l_i. Now let

$$\epsilon l_i=\alpha_i+\beta_i\epsilon. \tag{10}$$

Then if we expand ϵ^nC_n to first order in ϵ as we have done in Eq. (7), both in terms of α_i and β_i and in terms of invariants of $[SU(2)\otimes SU(2)]\times T_9$, we obtain the following relations:

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4=0;$$
$$\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2=\mathcal{P}_2;$$
$$\alpha_1^3+\alpha_2^3+\alpha_3^3+\alpha_4^3=-\tfrac{1}{2}\mathcal{P}_3;$$
$$\alpha_1^4+\alpha_2^4+\alpha_3^4+\alpha_4^4=\tfrac{1}{4}[3\mathcal{P}_2^2-2\mathcal{P}_4];$$
$$\beta_1+\beta_2+\beta_3+\beta_4=0;$$
$$\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3+\alpha_4\beta_4=-\tfrac{1}{2}(3\alpha_1+\alpha_2-\alpha_3-3\alpha_4);$$
$$\alpha_1^2\beta_1+\alpha_2^2\beta_2+\alpha_3^2\beta_3+\alpha_4^2\beta_4=-\tfrac{1}{2}(3\alpha_1^2+\alpha_2^2-\alpha_3^2-3\alpha_4^2);$$
$$\alpha_1^3\beta_1+\alpha_2^3\beta_2+\alpha_3^3\beta_3+\alpha_4^3\beta_4=-\tfrac{1}{3}((5\alpha_1^3+2\alpha_2^3-\alpha_3^3-4\alpha_4^3)+\tfrac{1}{2}(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4); \tag{11}$$

where $\mathcal{P}_2$, $\mathcal{P}_3$, and $\mathcal{P}_4$ are the invariants of $[SU(2)\otimes SU(2)]\times T_9$ and are given by

$$\mathcal{P}_2=A_{i\alpha}A_{i\alpha},$$
$$\mathcal{P}_3=\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}A_{i\alpha}A_{k\gamma},$$
$$\mathcal{P}_4=A_{i\alpha}A_{j\alpha}A_{j\beta}A_{i\beta}. \tag{12}$$

Let

$$\mathfrak{M}=(aI^2+bJ^2)3/4\mathcal{P}_2; \tag{13}$$

then Eq. (II) is satisfied if

$$A_{i\alpha}A_{j\alpha}=\tfrac{1}{3}\mathcal{P}_2\delta_{ij},\quad A_{i\alpha}A_{i\beta}=\tfrac{1}{3}\mathcal{P}_2\delta_{\alpha\beta}, \tag{14}$$

and $a+b=1$. Equation (14) relates the invariants of the algebra

$$\mathcal{P}_3^2=\tfrac{4}{3}\mathcal{P}_2^3,\quad \mathcal{P}_4=\tfrac{1}{4}\mathcal{P}_2^2.$$

Under these restrictions we see that Eqs. (11) have a solution

$$\alpha_1=3\alpha,\quad \alpha_2=\alpha_3=\alpha_4=-\alpha,$$
$$\beta_1=-\tfrac{3}{2},\quad \beta_2+\beta_3+\beta_4=\tfrac{3}{2}, \tag{15}$$

which gives the $(\infty,\lambda_2,\lambda_3)$ representation of SU(4) for the physically interesting irreducible unitary representation of $[SU(2)\otimes SU(2)]\times T_9$, where λ_2 and λ_3 are integers with $\lambda_2\geqslant\lambda_3\geqslant 0$.

To see the spin and isospin of the isobars we must reduce the representation $(\infty,\lambda_2,\lambda_3)$ to the irreducible representations of $SU(2)\otimes SU(2)$. A general method of this reduction has been studied by Hagen and Macfarlane.[6] An interesting case[7] is $(\infty,0,0)$, which gives the sequence of $SU(2)\otimes SU(2)$ representations with $t=j$, i.e., either $(0,0),(1,1),\cdots$, or $(\frac{1}{2},\frac{1}{2}),(\frac{3}{2},\frac{3}{2}),\cdots$, where (t,j) is a representation of $SU(2)\otimes SU(2)$. The mass spectrum is given by

$$M(t,j)=3R/4g^2[aj(j+1)+(1-a)t(t+1)], \tag{16}$$

where a is arbitrary. [This is because both terms yield the same form for $\Lambda_{\beta\alpha}$.]

To obtain the Lie group G of strong coupling $\{SU(2)\times T_3$ or $[SU(2)\otimes SU(2)]\times T_9\}$, we contracted a compact group $[SU(2)\otimes SU(2)$ or $SU(4)]$,

Table I. Groups of strong coupling (SC) and intermediate coupling (IC).

Group of invariance K	Group of SC G	Group of IC Compact	Group of IC Noncompact
SU(2)	SU(2) $\times T_3$	SU(2)⊗SU(2)	SL(2, C)
SU(2)⊗SU(2)	[SU(2)⊗SU(2)] $\times T_9$	SU(4)	SL(4, R)
SU(2)⊗SU(3)	[SU(2)⊗SU(3)] $\times T_{24}$	SU(6)	SL(6, R)
SU(4)	SU(4) $\times T_{15}$	SU(4)⊗SU(4)	SL(4, C)
SU(6)	SU(6) $\times T_{35}$	SU(6)⊗SU(6)	SL(6, C)

but the signs of some structure constants of the original group are irrelevant after the contraction (those structure constants that $\rightarrow 0$). If we choose opposite signs, the original Lie algebra would be noncompact. In other words, it is possible to perform the contraction from a noncompact group. In the previous examples, we could have used SL(2, C) and SL(4, R) instead of SU(2)⊗SU(2) and SU(4), respectively, although the irreducible unitary representations of these noncompact groups are hard to obtain in practice.

The strong coupling limiting process seems to be related to the mathematical concept of contraction. So, it would not be a bad guess that in case of finite coupling constant the relevant group may be a precontracted group, either compact or noncompact, and we may regard it as the "group of intermediate coupling" which presents the higher symmetry of elementary particles. Therefore, it is interesting to know these groups for the more complicated cases. In Table I we give the list of these groups.

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[3]There are solutions of other kinds by extending the algebra further. However, we shall discuss them on another occasion.

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[5]The derivation of relations (I) and (II) that was sketched above was for s-wave mesons. For p waves the same relations hold with the sole difference that in the relation between Δ and $\mathfrak{M}$, $\Delta = 2\mu\mathfrak{M}$, one must replace 2μ by R, where R is the cutoff radius. See reference 1.

[6]C. R. Hagen and A. J. Macfarlane, "Reduction of Representations of SU_{mn} with Respect to the Subgroup $SU_m \otimes SU_n$," Imperial College Report No. ICTP/64/76.

[7]Other interesting representations $(\infty, 1, 0)$ and $(\infty, 1, 1)$ have the following SU(2)⊗SU(2) decomposition: For $(\infty, 1, 0)$ either $(\frac{1}{2}, \frac{1}{2}) + (\frac{3}{2}, \frac{3}{2}) + \cdots + (\frac{1}{2}, \frac{3}{2}) + (\frac{3}{2}, \frac{5}{2}) + \cdots + (\frac{3}{2}, \frac{1}{2}) + (\frac{5}{2}, \frac{3}{2}) + \cdots$, or $(0, 0) + (1, 1) + \cdots + (0, 1) + (1, 2) + \cdots + (1, 0) + (2, 1) + \cdots$; and for $(\infty, 1, 1)$ either $(\frac{1}{2}, \frac{1}{2}) + 2(\frac{3}{2}, \frac{3}{2}) + 2(\frac{5}{2}, \frac{5}{2}) + \cdots + (\frac{1}{2}, \frac{3}{2}) + (\frac{3}{2}, \frac{5}{2}) + \cdots + (\frac{3}{2}, \frac{1}{2}) + (\frac{5}{2}, \frac{3}{2}) + \cdots$, or $(0, 0) + 2(1, 1) + 2(2, 2) + \cdots + (0, 1) + (1, 2) + \cdots + (1, 0) + (2, 1) + \cdots$.

NEUTRON-PROTON CHARGE-EXCHANGE SCATTERING IN THE BeV/c REGION

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Several years ago we measured the n-p elastic charge-exchange cross section at 2.04 and 2.85 BeV.[1] The interesting result from this work was the observation of a sharply peaked angular distribution with a half-width at half-maximum corresponding to a momentum transfer of 150 MeV/c. This is half the width of the pp diffraction peak at these energies. The width of the charge-exchange peak was found to be momentum-transfer invariant and, therefore, the 150-MeV/c width indicates that the difference between the $T = 1$ and $T = 0$ isotopic-spin-