NEW INSTABILITIES DUE TO HALL EFFECT

R. J. Hosking
Department of Mathematics and Astronomy, The University of Western Ontario, London, Ontario, Canada
(Received 7 July 1965)

It is now well known that the inclusion of finite conductivity in hydromagnetic-stability theory permits instabilities which do not occur in a perfectly conducting fluid. In the terminology of Furth, Killeen, and Rosenbluth[1] (hereafter referred to as FKR), fluid detachment from the magnetic-field lines permits the system to attain states of lower potential energy which are not topologically accessible in the absence of dissipation.

From the mathematical viewpoint, the omission of resistivity reduces the order of the differential equations involved. This may not be valid asymptotic procedure in certain regions, even though the dissipation may be very small (just as the kinematic viscosity remains important in the Prandtl boundary layer, in the limit of large Reynolds number). In the case of the resistive modes of FKR, $F(\vec{r}) = \vec{k}\cdot\vec{H}(\vec{r})$ defines a singular hypersurface near which resistivity is important, where k is the wave number of the instability and $\vec{H}(\vec{r})$ is the applied magnetic field.

On the other hand, it appears that the inclusion of ion-gyration effects has not proven destabilizing,[2-6] whether or not resistivity is also included. Coppi,[6] in particular, has shown that such effects may strongly reduce the growth rates of the resistive modes, such as those found by KFR.

Since introduction of the Hall effect does not raise the order of the differential equations involved in a stability analysis when finite conductivity is also included, it is not surprising that no new instabilities are found in this case. What is perhaps surprising is that these effects have not been noted to produce new instabilities when the fluid is assumed <u>infinitely</u> conducting. Roberts and Taylor[4] did note a new mode, but it was a high-frequency oscillation. We shall consider a simple situation in which it appears that the Hall effect produces an aperiodic instability in a wave-number band which was previously stable.

The situation chosen is similar to that first considered by Kruskal and Schwarzschild,[7] where an infinitely conducting plasma occupies the half-space $0<z<\infty$ and is supported against gravity by magnetic pressure due to a jump in field strength at the plasma boundary $z=0$. [(x,y,z) are Cartesian coordinates.] In our case we shall assume that the plasma exhibits a significant Hall effect and that the seed field $\vec{H}_0=(H_0,0,0)$ is uniform throughout all space. The equilibrium equations are

$$\vec{\nabla}p_0=\rho\vec{g},$$
$$\vec{E}_0+(1/ne)\vec{\nabla}p_{e0}=0,$$

where $\vec{g}=(0,0,-g)$ is the constant (vertical) gravitational field, p_0 is the equilibrium fluid pressure, p_{e0} is the equilibrium electron pressure, n is the electron number density, e is the electronic charge, and $\vec{E}_0$ is the equilibrium electric field. Assuming equal ion and electron temperatures so that $p_i=p_e=\frac{1}{2}p$, one may consider that the hydrostatic pressure required to support the plasma equilibrium is supplied by the electric field.

The perturbed physical quantities are assumed to be of the form

$$q(x,z,t)=q_0(z)+q_1(z)\exp(\omega t+ikx),$$

so that only perturbations propagated along the seed field and which bend the lines of force are considered. The perturbation equations are then

$$\rho\omega\vec{u}_1+\vec{\nabla}\pi_1=i\mu kH_0\vec{H}_1, \tag{1}$$

where

$$\pi_1=p_1+\mu H_0H_{1z};$$
$$\omega\vec{H}_1=ikH_0\vec{u}_1-i(kH_0/ne)\vec{\nabla}\times\vec{H}_1; \tag{2}$$

$$\vec{\nabla}\cdot\vec{u}=\vec{\nabla}\cdot\vec{H}=0; \tag{3}$$

in the plasma, and

$$\vec{\nabla}\times\vec{H}=\vec{\nabla}\cdot\vec{H}=0 \tag{4}$$

in the vacuum, where $\vec{u}$ is the fluid velocity and the unity subscript denotes a perturbation quantity.

At the perturbed boundary, one has continuity of both total pressure and normal field component, together with a condition derived from

Ohm's law in the assumed form[2]

$$\vec{E}+\mu\vec{u}\times\vec{H}-\frac{\mu}{ne}\vec{\delta}\times\vec{H}+\frac{1}{ne}\vec{\nabla}p_e=0. \qquad (5)$$

It follows after differentiation with respect to time that

$$p_1-\rho g\frac{u_{1z}}{\omega}=0, \qquad (6)$$

$$H_{1x}|_{z=0^+}=H_{1x}|_{z=0^-}, \qquad (7)$$

$$H_{1z}|_{z=0^+}=H_{1z}|_{z=0^-}. \qquad (8)$$

In addition, the perturbations must be bounded as $|z|\to\infty$.

Elimination of $\vec{u}_1$ between (1) and (2) yields

$$\frac{ikH_0}{ne}\vec{\nabla}\times\vec{\nabla}\times\vec{H}_1+\left(\omega+\mu\frac{k^2H_0^2}{\rho\omega}\right)\vec{\nabla}\times\vec{H}_1=0. \qquad (9)$$

Hence, in the plasma

$$H_{1z}=C_1e^{-kz}+C_2e^{-\varphi z},$$

where (without loss of generality) φ is the root with its positive real part being

$$\varphi^2=k^2+\left[\left(\omega+\mu\frac{k^2H_0^2}{\rho\omega}\right)\frac{ne}{kH_0}\right]^2, \qquad (10)$$

while in the vacuum

$$H_{1z}=C_3e^{kz}.$$

With the help of the x component of (1) together with (3) and (4), from the boundary conditions (6)-(8) one has a secular equation which reduces to the dispersion relation

$$\frac{\varphi}{k}=\frac{gk/\omega-\omega+2\mu(k^2H_0^2/\rho\omega^2)(gk/\omega)}{-gk/\omega+\omega+2\mu k^2H_0^2/\rho\omega}=\frac{N}{D}. \qquad (11)$$

The influence of the Hall effect may be seen directly from dispersion relation (11). The perfect conductivity limit of Kruskal and Schwarzschild[7] corresponds to ion gyrofrequency $\omega_i=e\mu H_0/m_i\approx ne\mu H_0/\rho\equiv\infty$, where m_i is the ionic mass. At this limit, $\varphi=\infty$ from (10), so that

$$D=-\frac{gk}{\omega}+\omega+2\mu\frac{k^2H_0^2}{\rho\omega}=0,$$

or

$$\omega^2=gk\left(1-\frac{2\mu H_0^2}{\rho}\frac{k}{g}\right);$$

there is stability for $k>k_c=\rho g/2\mu H_0^2$. When ω_i is large but finite, however, for small real ω,

$$\frac{\varphi}{k}\to\frac{\mu H_0^2}{\rho}\frac{ne}{H_0}\frac{1}{\omega},$$

$$N\to\frac{\mu H_0^2}{\rho}2k^3g\frac{1}{\omega^3}$$

$$D\to\left(2k\frac{\mu H_0^2}{\rho}-g\right)\frac{k}{\omega},$$

so that

$$\omega\approx\frac{H_0}{ne}\frac{\rho}{\mu H_0^2}\frac{2k^2g}{2k-\rho g/\mu H_0^2},$$

and there is stability for $k<k_c=\rho g/2\mu H_0^2$.

The introduction of the Hall effect thus permits aperiodic instabilities in just that wave-number band for which there is complete stability in the "frozen-in field" approximation. Reference to Eq. (9) shows that the inclusion of the Hall effect means that the perturbed current density $\vec{\delta}_1=\vec{\nabla}\times\vec{H}_1$ is not zero throughout all space as it would otherwise be, but falls off exponentially with z. The current skin depth defines the region in which the Hall effect is important—where derivatives of the current components and hence the first term of (9) is significant.

In conclusion, it should be emphasised that resistive modes are precluded, since Eq. (5) is the form of Ohm's law adopted. Also, the viscositylike finite gyroradius term in the equation of motion has been neglected, and this may contribute additional effects.[4]

[1]H. P. Furth, J. Killeen, and M. N. Rosenbluth, Phys. Fluids 6, 459 (1963).

[2]R. J. Tayler, Nucl. Fusion, Suppl. Pt. 3, 887 (1962).

[3]M. N. Rosenbluth, N. A. Krall, and N. Rostoker, Nucl. Fusion, Suppl. Pt. 1, 143 (1962).

[4]K. V. Roberts and J. B. Taylor, Phys. Rev. Letters 8, 197 (1962).

[5]R. J. Tayler, J. Nucl. Energy, Pt. C 5, 345 (1963).

[6]B. Coppi, Phys. Fluids 7, 1501 (1964).

[7]M. Kruskal and M. Schwarzschild, Proc. Roy. Soc. (London) A223, 348 (1954).