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NONLINEAR TWO-FLUID EQUATIONS FOR A SUPERCONDUCTOR

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The recent interest in dissipative effects in type-II superconductors has pointed up the need for two-fluid equations which are nonlinear in the velocities and describe the motion of the normal electrons without the contradiction encountered in the London theory.[1] The purpose of the present Letter is to give the phenomenological equations of motion which are obtained from Eckart's[2] variational principle as applied to the two-fluid model of a superconductor. The derivation follows closely that used by Zilsel[3] to obtain the two-fluid equations for liquid-helium II, but with the addition of the electromagnetic terms; it differs from Cook's[4] derivations of the London equations from Eckart's principle in that it includes the normal electrons. The resulting equations yield a source term for the normal electrons and the explicit form of the osmotic-pressure term suggested by London[1] and recently applied to extreme type-II superconductors by Vijfeijken and Staas.[5]

Eckart's principle assumes that the dynamics of a continuous system are such that

$$\delta\int_{t_1}^{t_2}\int \mathcal{L}\, d^3r dt = 0, \tag{1}$$

where $\mathcal{L}$ is the Lagrangian density in a given region whose boundaries are fixed. There is considerable evidence that this variational principle, which is more restrictive than Hamilton's principle, gives the correct description of superfluids. The Lagrangian density is assumed to be[6]

$$\mathcal{L} = \tfrac{1}{2}x\rho\vec{V}_n^{\,2} + \tfrac{1}{2}(1-x)\rho\vec{V}_s^{\,2} + \frac{e}{mc}x\rho\vec{V}_n\cdot\vec{A} + \frac{e}{mc}(1-x)\rho\vec{V}_s \cdot\vec{A} - \frac{e}{m}\rho\Phi - \rho U(\rho,S,x) + \frac{\epsilon}{8\pi}\left[-\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right]^2 - \frac{1}{8\pi\mu}(\nabla\times\vec{A})^2. \tag{2}$$

Here $x=\rho_n/\rho$ and $1-x=\rho_s/\rho$ are the normal and superfluid mass fractions and $\vec{V}_n$ and $\vec{V}_s$ are the normal and superfluid velocities. S is the entropy and U is the internal energy, both per unit mass. Φ and A are the scalar and vector potential and e/m is the charge-to-mass ratio of the carriers. Equation (1) is to be restricted by conservation of mass (which for electronic carriers also implies conservation of charge) and conservation of entropy:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho[x\vec{V}_n + (1-x)\vec{V}_s] = 0, \tag{3}$$

$$\frac{\partial(\rho S)}{\partial t} + \nabla\cdot[\rho S\vec{V}_n] = 0. \tag{4}$$

Equation (4) also contains the assumption that entropy is carried only by the normal electrons. Conservation of entropy is slightly relaxed later by including a linear frictional force in the normal fluid equation. Lin's[7] constraint, which is necessary for a classical fluid, and which Whitlock[8] used to derive the one-fluid plasma equations, is assumed not to apply to a quantum fluid where the particle trajectory is not a defined concept. The variational principle with these two constraints is therefore

$$\delta\int_{t_1}^{t_2}\int\{\mathcal{L} - \alpha(\partial\rho/\partial t + \nabla\cdot\rho[x\vec{V}_n + (1-x)\vec{V}_s]) - \beta[\partial(\rho S)/\partial t + \nabla\cdot\rho S\vec{V}_n]\}d^3r dt = 0. \tag{5}$$

Variation with respect to $\rho, S, x, \vec{V}_n, \vec{V}_s, \Phi, \vec{A}$ lead to the following relations:

$$\delta\rho:\ \tfrac{1}{2}x\vec{V}_n^{\,2} + \tfrac{1}{2}(1-x)\vec{V}_s^{\,2} + \frac{e}{mc}x\vec{V}_n\cdot\vec{A} + \frac{e}{mc}(1-x)\vec{V}_s \cdot\vec{A} + \frac{e}{m}\Phi - U - \frac{1}{\rho}P + \partial\alpha/\partial t + [x\vec{V}_n + (1-x)\vec{V}_s] \cdot\nabla\alpha + S(\partial\beta/\partial t) + S\vec{V}_n\cdot\nabla\beta = 0, \tag{6}$$

$$\delta s:\ -T + \partial\beta/\partial t + \vec{V}_n\cdot\nabla\beta = 0, \tag{7}$$

$$\delta x:\ \tfrac{1}{2}(\vec{V}_n{}^2-\vec{V}_s{}^2)+\frac{e}{mc}(\vec{V}_n-\vec{V}_s)\cdot\vec{A}-(\partial V/\partial x)_{\rho,S}+(\vec{V}_n-\vec{V}_s)\cdot\nabla\alpha=0, \quad (8)$$

$$\delta\vec{V}_n:\ \vec{V}_n+(e/mc)\vec{A}+\nabla\alpha+(S/x)\nabla\beta=0, \quad (9)$$

$$\delta\vec{V}_s:\ \vec{V}_s+(e/mc)\vec{A}+\nabla\alpha=0, \quad (10)$$

$$\delta\Phi:\ 4\pi e\rho/m+\epsilon\nabla\cdot[\nabla\Phi+(1/c)\partial A/\partial t]=0, \quad (11)$$

$$\delta A:\ (4\pi e\rho/mc)[x\vec{V}_n+(1-x)\vec{V}_s]-(1/\mu)(\nabla\times\nabla\times\vec{A})-\epsilon\partial/\partial t[\nabla\Phi+(1/c)\partial\vec{A}/\partial t]=0. \quad (12)$$

Equations (11) and (12) lead to Maxwell's equations, and from the curl of Eq. (10) we obtain the first London equation

$$\nabla\times\vec{V}_s-(e/mc)\vec{B}=0.$$

If we eliminate the Lagrange multipliers, α and β, from Eqs. (6) through (10), we obtain the equation of motion of the superfluid,

$$\partial\vec{V}_s/\partial t+\nabla(\vec{V}_s{}^2/2)=-\nabla p/\rho+S\nabla T+(x/2)\nabla|\vec{V}_n-\vec{V}_s|^2-(e/m)[\nabla\Phi+(1/c)\partial\vec{A}/\partial t]. \quad (13)$$

The equation for the normal fluid is also obtained from Eqs. (7), (9), and (13), with the additional assumption of a frictional force, Ohm's law. Following Zilsel, the normal fluid equation contains a source term Γ, which is the rate of production of normal fluid mass/unit volume.

$$\partial\vec{V}_n/\partial t+\nabla(\vec{V}_n{}^2/2)-\vec{V}_n\times\nabla\times\vec{V}_n=-\nabla p/\rho-[(1-x)/x]\nabla T-[(1-x)/2]\nabla|\vec{V}_n-\vec{V}_s|^2-(\vec{V}_n-\vec{V}_s)(\Gamma/\rho x)-(e/m)\times[\nabla\Phi+(1/c)\partial\vec{A}/\partial t]-x\gamma\vec{V}_n. \quad (14)$$

Actually, the pressure p can be disregarded as compared with the electric field for distances large compared with the Debye screening distance.[9]

Equations (13) and (14) avoid the inconsistency of the London treatment of the normal electrons. Thus the London equations for $\partial\vec{V}_s/\partial t=0$ give an electric field $\vec{E}=(m/e)\nabla(\vec{V}_s{}^2/2)$, which (assuming the normal electrons obey Ohm's law $\vec{J}_n=\sigma_n\vec{E}$) implies dissipation whenever there is a gradient in the superfluid current density, as in screening currents in the penetration depth. That there actually is no such dissipation follows from the existence of persistent currents and the Meissner effect. This contradiction is removed by the terms in Eqs. (13) and (14) containing $\nabla|\vec{V}_n-\vec{V}_s|^2$, which provide a balancing force to prevent gradients of the superfluid velocity field from accelerating the normal electrons.

Equations (13) and (14) also imply the possibility of thermal counterflow, second sound, a thermoelectric potential analogous to the fountain pressure of liquid-helium II, and a Hall effect of the normal electrons. Presumably these effects could only be observed if the scattering of normal electrons by the lattice is very small.

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