INTERACTION OF "SOLITONS" IN A COLLISIONLESS PLASMA AND THE RECURRENCE OF INITIAL STATES

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(Received 3 May 1965)

We have observed unusual nonlinear interactions among "solitary-wave pulses" propagating in nonlinear dispersive media. These phenomena were observed in the numerical solutions of the Korteweg-deVries equation

$$u_t + uu_x + \delta^2 u_{xxx} = 0. \tag{1}$$

This equation can be used to describe the one-dimensional, long-time asymptotic behavior of small, but finite amplitude: shallow-water waves,[1,2] collisionless-plasma magnetohydrodynamic waves,[2] and long waves in anharmonic crystals.[3,4] Furthermore, the interaction and "focusing" in space-time of the solitary-wave pulses allows us to give a phenomenological description (some aspects of which we can already explain analytically) of the near recurrence to the initial state in numerical calculations for a discretized weakly-nonlinear string made by Fermi, Pasta, and Ulam (FPU).[4,5]

Spatially periodic numerical solutions of the Korteweg-deVries equation were obtained with a scheme that conserves momentum and almost conserves energy.[6] For a variety of initial conditions normalized to an amplitude of 1.0 and for small δ^2, the computational phenomena observed can be described in terms of three time intervals. (I) Initially, the first two terms of Eq. (1) dominate and the classical overtaking phenomenon occurs; that is, u steepens in regions where it has a negative slope. (II) Second, after u has steepened sufficiently, the third term becomes important and serves to prevent the formation of a discontinuity. Instead, oscillations of small wavelength (of order δ) develop on the left of the front. The amplitudes of the oscillations grow and finally each oscillation achieves an almost steady amplitude (which increases linearly from left to right) and has a shape almost identical to that of an individual solitary-wave solution of (1). (III) Finally, each such "solitary-wave pulse" or "soliton" begins to move uniformly at a rate (relative to the background value of u from which the pulse rises) which is linearly proportional to its amplitude. Thus, the solitons spread apart. Because of the periodicity, two or more solitons eventually overlap spatially and interact nonlinearly. Shortly after the interaction, they reappear virtually unaffected in size or shape. In other words, solitons "pass through" one another without losing their identity. Here we have a nonlinear physical process in which interacting localized pulses do not scatter irreversibly.

It is desirable to elaborate the concept of the soliton, for it plays such an important role in explaining the observed phenomena. We seek stationary solutions of (1) in a frame moving with velocity c. We substitute

$$u = U(x-ct)$$

into (1) and obtain a third-order nonlinear ordinary differential equation for u. This has periodic solutions representing wave trains, but to explain the concept of a soliton we are interested in a solution which is asymptotically constant at infinity ($u = u_\infty$ at $x = \pm\infty$). The result[7] of such a calculation is

$$u = u_\infty + (u_0 - u_\infty)\,\mathrm{sech}^2[(x-x_0)/\Delta], \tag{2}$$

where u_0, u_∞, and x_0 are arbitrary constants and

$$\Delta = \delta[(u_0 - u_\infty)/12]^{-1/2}, \tag{3}$$

and

$$c = u_\infty + (u_0 - u_\infty)/3. \tag{4}$$

Thus, the larger the pulse amplitude and the smaller δ, the narrower is the pulse. The surprising thing is that these pulses, which are strict solutions only when completely isolated, can exist in close proximity and interact without losing their form or identity (except mo-

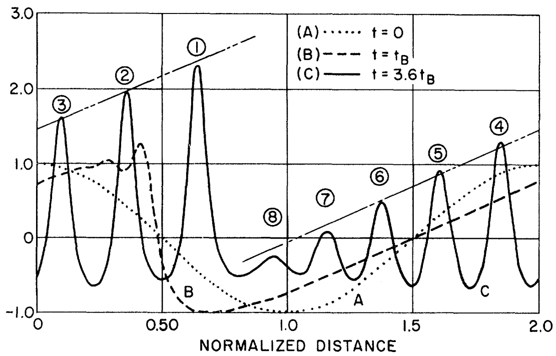

FIG. 1. The temporal development of the wave form $u(x)$.

mentarily while they "overlap" substantially).

The numerical calculations in which these phenomena were observed were made starting with $\delta = 0.022$ and the periodic initial condition

$$u|_0 = \cos\pi x. \tag{5}$$

Thus, initially, $\max|\delta^2 u_{xxx}|/\max|uu_x| = 0.004$ so the third term can be neglected and we are dealing with the equation $u_t + uu_x = 0$. Its solution is given by the implicit relation

$$u = \cos\pi(x - ut), \tag{6}$$

and we find that u tends to become discontinuous at $x = \frac{1}{2}$ and $t = T_B = 1/\pi$, the breakdown time. Figure 1, curve A, gives the initial condition, and curve B shows the function at T_B. The slight oscillatory structure for $x < \frac{1}{2}$ is due to the third derivative which we have neglected in arriving at the approximate solution (6). Curve C at $t = 3.6T_B$ shows a train of solitons (numbered 1-8), which have developed from the oscillations. A so-far unexplained property of these solutions is the linear variation of the amplitude of the largest pulses. Table I gives the amplitudes of the pulses, their observed and calculated widths [Eq. (3)], and their observed and calculated velocities [Eq. (4)]. We note that the calculated and observed widths and velocities of the first seven solitons are in very good agreement.

Figure 2 gives the space-time trajectories of the solitons. The vertical axis is normalized in terms of the recurrence time T_R ($T_R = 30.4T_B$ for this computation), which is the time it would take all the solitons to overlap or "focus" at a common spatial point. The diagram at the right of Fig. 2 shows the amplitude of soliton no. 1 (horizontally) versus time (vertically). The observed velocity of each soliton given in the table is calculated as the slope of the straight line drawn tangent to its trajectory over the time interval $0.0975T_R$ to $0.133T_R$.

When solitons of very different amplitude approach, their trajectories deviate from straight lines (accelerate) as they "pass through" one another. During the overlap time interval the

Table I. Soliton properties—observed[a] and calculated values ($\delta = 0.022, t = 3.6T_B$).

Pulse no.	Amplitudes (observed) $-u_\infty$	u_0-u_∞	Width (Δ) observed	calculated	Velocity (c) observed	calculated
1	325	1739	0.0455	0.0456	227	254
2	401	1597	0.0475	0.0476	110	131
3	491	1485	0.0492	0.0493	0	4
4	544	1318	0.0522	0.0516	−99	−105
5	574	1115	0.0567	0.0568	−169	−202
6	584	885	0.0636	0.0639	−273	−289
7	558	610	0.0769	0.0767	−361	−354
8	453	302	0.099	0.109	−443	−353

[a]The observed quantities (excluding c) are obtained from the numerical values of u and u_{xx} at $t = 3.6T_B$ (Fig. 1, curve C). u_0 is the observed maximum of each pulse; u_0-u_∞ (and therefore u_∞) and $\Delta_{\rm obs}$ are obtained from the minimum value of u_{xx}:

$$\Delta_{\rm obs} = (24\delta^2 |\min u_{xx}|_{x=x_0}|)^{1/4},$$

$$u_0 - u_\infty = 12\delta^2/\Delta_{\rm obs}^2.$$

The observed values of c are obtained by measuring slopes on Fig. 2 at $t = 3.5T_B = 0.115T_R$, as described below. The calculated values of Δ are obtained from Eq. (3) with $\delta = 0.022$ and (u_0-u_∞) obtained from column 3. The calculated values of c are obtained from (4) with u_∞ and (u_0-u_∞) obtained from columns 2 and 3.

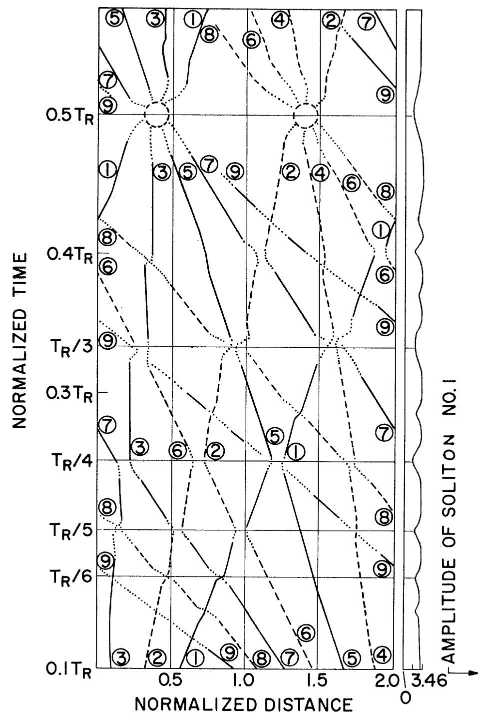

FIG. 2. Soliton trajectories on a space-time diagram beginning at $t = 0.1T_R = 3.04\,T_B$. The diagram at the right shows the variation of the amplitude of soliton no. 1 as a function of time.

joint amplitude of the interacting solitons decreases (in contradistinction to what would happen if two pulses overlapped linearly). This is evident at $t = T_R/6$ where solitons 1 and 7, 2 and 8, and 3 and 9 overlap. When the amplitudes of approaching solitons are comparable they seem to exchange amplitudes and therefore velocities. They do not have to approach very close to one another for this "transition" to occur. This is evident at $t = T_R/4$ where solitons 1 and 5, 2 and 6, 3 and 7, and 4 and 8 overlap (and similarly at $t = T_R/3$, etc.). In general, when solitons approach we have replaced the solid lines (odd-numbered solitons) and dashed lines (even-numbered solitons) by dots, as it is not yet clear how to describe what happens during a close interaction. At $t = 0.5T_R$ all the odd solitons overlap at $x = 0.385$ and all the even ones at $x = 1.385$, and because of the nonlinear interaction one cannot follow the crests, and so the regions are circled. The waveform of u which results is mostly composed of, and has the form of, the second harmonic of the initial waveform.

In conclusion, we should emphasize that at T_R all the solitons arrive almost in the same phase and almost reconstruct the initial state through the nonlinear interaction. This process proceeds onwards, and at $2T_R$ one again has a "near recurrence" which is not as good as the first recurrence. Tuck,[8] at the Los Alamos Scientific Laboratories, observed this phenomenon as well as eventual "superrecurrences" in calculations for a similar problem. We can understand these phenomena in terms of soliton interactions. For $t > T_R$ the successive focusings get poorer due to solitons arriving more and more out of phase with each other and then eventually gets better again when their phase relationship changes. Furthermore, because the solitons are remarkably stable entities, preserving their identity through numerous interactions, one would expect this system to exhibit thermalization (complete energy sharing among the corresponding linear normal modes) only after extremely long times, if ever.

The authors would like to thank G. S. Deem for assistance in programming and reducing the numerical data.

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[1]J. J. Stoker, *Water Waves* (Interscience Publishers, Inc., New York, 1957). See Secs. 10.9 through 10.11 and Eq. (10.11.20).

[2]C. S. Gardner and G. K. Morikawa, Courant Institute of Mathematical Sciences Report No. NYO 9082, 1 May 1960 (unpublished).

[3]N. J. Zabusky, in *Proceedings of the Conference on Mathematical Models in the Physical Sciences*, edited by Stefan Drobot (Prentice-Hall, Inc., New York, 1963), p. 99.

[4]M. D. Kruskal, "Progress on the Fermi, Pasta, Ulam Problem" (to be published).

[5]E. Fermi, J. R. Pasta, and S. Ulam, Los Alamos Scientific Laboratory Report No. LA-1940, May 1955 (unpublished). See reference 3 for a review of the problem.

[6]We restrict ourselves to solutions of (1) periodic in x with period 2 so that we need only consider the interval $0 \leq x < 2$ with periodic (cyclic) boundary conditions. For numerical purposes we replaced (1) with

$$u_i^{j+1} = u_i^{j-1} - \tfrac{1}{3}(k/h)(u_{i+1}^{\,j} + u_i^{\,j} + u_{i-1}^{\,j})(u_{i+1}^{\,j} - u_{i-1}^{\,j}) - (\delta^2 k/h^3)(u_{i+2}^{\,j} - 2u_{i+1}^{\,j} + 2u_{i-1}^{\,j} - u_{i-2}^{\,j}),$$

$$i = 0,\ 1, \cdots, 2N-1,$$

where a rectangular mesh has been used with temporal and spatial intervals of k and $h = 1/N$, respectively. That is, the function $u(x, t)$ is approximated by $u_i^{\,j} = u(ih, jk)$. In performing the calculations we used periodic (cyclic) boundary conditions $u_i^j = u_{i+2N}^j$. The momentum

$$\sum_{i=0}^{2N-1} u_i^j$$

is identically conserved, for if one sums both sides of the equation with respect to i, then the quantities multiplied by k vanish identically. The energy

$$\sum_{i=0}^{2N-1} \tfrac{1}{2}(u_i^j)^2$$

is almost conserved, that is, the above quantity is invariant if we neglect terms

$$(k^2/6)\sum_{i=0}^{2N-1} u_i(u_i)_{ttt} + O(k^4).$$

This is evident if we replace $(u_i^{j+1} - u_i^{j-1})/2k$ by $\partial u_i/\partial t$, multiply through by u_i, and sum. In practice, those runs which are numerically stable conserve the energy to five significant figures. The details of numerical computation and analysis will be published in the near future.

[7]See reference 2, p. 18 for a similar calculation with $u_\infty = 0$.

[8]J. Tuck, private communication.