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10 EXACT GRAVITATIONALLY-CONSERVED QUANTITIES*

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In ordinary electromagnetic theory, total charge remains exactly constant for any system surrounded by source-free space. This is a direct consequence of Maxwell's equations for the free-space region, whereas dipole and higher moments can be carried away by electromagnetic radiation. In gravitational theory, analogs of this conserved charge would, at first sight, appear to be mass, momentum, and angular momentum, as they are indeed exactly conserved in the linearized Einstein theory, for regions surrounded by empty space.[1,2] However, in the full nonlinear theory mass and momentum can be changed (i.e., "carried away") by gravitational radiation.[3-5] It has therefore been commonly believed that there are no (nontrivial) exactly conserved (covariant) gravitational quantities in general relativity. But it turns out that, quite unexpectedly, there is a set of 10 geometrical quantities, defined for asymptotically flat space-times, which have a quadrupole structure, and whose values cannot be altered in any way by gravitational radiation.

For a general asymptotically flat space-time[3-6] the quantities can be expressed as integrals at infinity on any null hypersurface $\mathfrak{N}$ which diverges to infinity. Using the notation of Newman-Unti,[5,7] these expressions are

$$Q_m = \int\int \Psi_0{}^1 {}_2Y_{2,m} \sin\theta d\varphi d\theta,$$
$$m = -2, -1, 0, 1, 2. \qquad (1)$$

Here $\Psi_0{}^1$ is defined by $-C_{\mu\nu\rho\sigma}l^\mu m^\nu l^\rho m^\sigma = \Psi_0 = \Psi_0{}^0 r^{-5} + \Psi_0{}^1 r^{-6} + O(r^{-7})$, where r is an affine (or luminosity) parameter, with $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ as standard null tetrad, on $\mathfrak{N}$, such that $r^2(d\theta^2 + \sin^2\theta d\varphi^2)$ defines, for large r, the metric of the "sphere at infinity" on $\mathfrak{N}$ ($\Psi_0{}^0$, $\Psi_0{}^1$ are independent of r). The quantities ${}_2Y_{2,m}$ are effectively "tensor" spherical harmonics[2] defined in terms of the ordinary harmonics $Y_{2,m}$ by

$${}_2Y_{2,m} = \left(\sin\theta\frac{\partial}{\partial\theta} + i\frac{\partial}{\partial\varphi}\right)\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi}\right)Y_{2,m}.$$

The real and imaginary parts of the five complex quantities Q_m are the required conserved quantities. Together they constitute a representation $D(2,0)$ of the homogeneous Lorentz group [of conformal motions of the (θ,φ) sphere].[4] They are conserved in the sense that if space-time is empty near infinity and if $\mathfrak{N}'$ is any other null hypersurface diverging to infinity, then the new values of Q_m will be the same as the old, the θ, φ for $\mathfrak{N}'$ being carried over from those for $\mathfrak{M}$ in the standard way, i.e., by the "supertranslation" of $\mathfrak{M}$ into $\mathfrak{M}'$.[3,4] Thus, the Q_m also form a $D(2,0)$ representation[8] of the full Bondi-Metzner-Sachs group.[3,4,9] This assumes that $\mathfrak{M}$ and $\mathfrak{M}'$ both open out into the future, so that outgoing gravitational radiation might escape between them. We could also define 10 additional conserved quantities by using (1) on null hypersurfaces which open out into the past. These would, in general, be different from the above Q_m, but the fact that they are conserved would seem to be of lesser interest, for retarded fields.[6]

It may be asked what is the interpretation of (1) in the linearized theory. Again the Q_m would be conserved, but in this case we must interpret (1) as representing the presence of incoming radiation (assuming $\mathfrak{M}$ opens into the future). It can be shown that (1) in fact vanishes for all retarded fields in the linearized theory. It would be natural, therefore, to attempt to interpret (1) in terms of incoming radiation in the full theory also. However, it is here that the full theory presents us with an essentially new situation. As it stands, (1) does not represent incoming radiation alone. This is brought out most clearly by an examination of the static and stationary vacuum fields. In such cases incoming radiation is absent (according to any reasonable definition!), but the Q_m can be expressed as certain combinations of mass, dipole, and quadrupole moments. For, if in (1) we replace $\Psi_0^{\,1}$ by

$$-5\Psi_0^{\,0}\Psi_2^{\,0}+(10/3)(\Psi_1^{\,0})^2, \qquad (2)$$

the field being stationary, and we choose $\mathfrak{M}$ (as we now may) so that the leading term σ^0 in the shear vanishes, then it turns out that the same Q_m are still obtained. Here $\Psi_2^{\,0}$ is real, constant, and represents the total mass; $\Psi_1^{\,0}, \Psi_0^{\,0}$ are complex functions of θ and φ describing, respectively, dipole and quadrupole structure of the source.[5,2] [Expression (2) has a quadrupole angular structure and is now equivalent to the Q_m.] In the case of the Weyl-Levi-Civita solutions only Q_0 survives and is real, but it does not vanish in general, i.e., unless the particular combination of moments given by (2) vanishes.

This leads to a remarkable state of affairs. Suppose an initially asymmetric body [with (2) nonvanishing] becomes spherically symmetric at a later time. In the meantime it must radiate gravitationally and the quantity (2) may change, but if the system is ever to become stationary again, the combination of moments (2) would have to return to its original value! It appears that we must interpret this in terms of back-scattering of the gravitational waves. Thus, during the radiative period, (1) represents not only a multipole structure of the source, but, in addition, a part apparently corresponding to incoming (i.e., scattered) radiation which would destroy the stationary nature of the field of the body—unless and until the original values of (2) are regained! That incoming radiation can contribute to the Q_m is clear from the results of the linear theory.

It should be mentioned here that while the Q_m determine a selection rule for purely gravitational transitions between stationary states, in the presence of electromagnetism (or neutrinos) the Q_m are no longer conserved. (It is conceivable that this has relevance to gravitational collapse, since a transition which cannot be accomplished by gravitational radiation alone might take place electromagnetically!) However, when electromagnetism is present, six new exactly conserved quantities emerge, namely the analogs of (1) for the Maxwell field. They have a dipole structure and depend on the r^{-4} part of the Maxwell field; they apparently do not necessarily vanish for stationary fields and they are conserved in asymptotically flat space-times if the Einstein-Maxwell equations hold near infinity. (Also there are four or eight conserved quantities for neutrino fields.)

Full details of these results, with proofs, will appear elsewhere.

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FUNDAMENTAL THEOREM IN QUANTUM OPTICS

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In optics we are concerned with a source, the field, and the interaction of the field with given systems. These systems usually include a detector and a—generally unavoidable—dissipation mechanism. For simplicity of terminology, we refer to all of these systems as the "detector"; the interaction of the field with the "detector" is the interaction under consideration. The concepts and formalism of quantum optics are greatly simplified by the following theorem:

All sources on which the effect of the "detector" is negligible may be treated as classical sources in the interaction under consideration. As a corollary of this theorem, we have:

The field acting on a quantum-mechanical system—when the sources are of the above type—consists of the superposition of a classical field and the "vacuum" field. The term "vacuum" field refers either to the true vacuum field in the absence of dissipation, or to the field due to the zero-point (and thermal, if pertinent) fluctuations of the dissipation mechanism, which replaces the true vacuum field,[1,2] in the presence of dissipation. It should be noted that the total field referred to in the corollary is fully quantum mechanical.

The proof of the theorem is based on the connection between the quantum-mechanical (as distinct from classical) properties of a system and the measurability of these properties. The essential difference between a quantum-mechanical and classical system may be described by the statement that a (ideal) measurement will disturb the former but not the latter. If certain measurements performed on a system do not disturb it, the behavior of the system with respect to these measurements may be regarded as prescribed, and described by a c number. (This description may, of course, be statistical, utilizing a statistical—but not quantum-mechanical—ensemble.) Our hypothesis, that the effect of the "detector" on the source is negligible, implies that the disturbance of the source by measurements made by the "detector" are negligible, and therefore the nonclassical properties of the source, as far as the interaction under consideration is concerned, are negligible.

Specifically, let us consider a mode of the electromagnetic field to which both the source and detector are coupled. Let us also separate (conceptually) the dissipation mechanism from the "detector" and consider it, henceforth, as a separate system. (We assume that this is the only significant dissipation associated with the mode under consideration; that is, the source contribution to the dissipation is negligible.) With some approximations based mainly on recognition of the fact that the usual dissipation mechanism, described by a dissipation constant β, is a linear system, it has been shown[1,2] that the coordinate and momentum of the radiation oscillator of frequency ω are given (for $\beta \ll \omega$) by

$$q(t)=\int_{-\infty}^{t}dt_1[\alpha S(t_1)+\gamma D(t_1)+F(t_1)] \times e^{-\frac{1}{2}\beta(t-t_1)}\cos\omega(t-t_1), \quad (1)$$

$$p(t)=-\int_{-\infty}^{t}dt_1[\alpha S(t_1)+\gamma D(t_1)+F(t_1)] \times e^{-\frac{1}{2}\beta(t-t_1)}\sin\omega(t-t_1), \quad (2)$$

where α and γ are coupling constants, S, D, and F are the dynamical variables through which the source, detector, and dissipation mechanism, respectively, couple to the mode,[3] and the units of q and p are dimensionless. We can write

$$q(t)\equiv q_S(t)+q_D(t)+q_F(t), \quad (3)$$

$$p(t)\equiv p_S(t)+p_D(t)+p_F(t), \quad (4)$$

in obvious notation. Now q_S, q_D, and q_F depend only on the dynamical variables of the respective systems to which they refer. (β is assumed to be a known constant.) Since non-