CHARACTERISTIC NONINVARIANCE GROUPS OF DYNAMICAL SYSTEMS*

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In recent years group theory has played an increasingly important role in particle physics. The orthodox notion of a symmetry group relates to an exact invariance group of the Hamiltonian, which entails conservation laws and selection rules. To a large extent the isospin group SU(2) is considered in this orthodox way. However, more recent studies of symmetries in particle physics have shown the relevance of larger symmetry groups such as SU(3), SU(4), and SU(6). These groups contain the isospin group as a subgroup, but are only "approximate" symmetries of the Hamiltonian. The most important role played by these groups is in the organization of isospin multiplets into super-multiplets. It turns out, however, that in some special models these approximate symmetries can be related to the dynamics of the system.[1,2] It is therefore relevant to investigate the question of characterizing the dynamical system in terms of its approximate symmetries.[3,4]

In this note we consider the general nature of the supermultiplet or "approximate" symmetry structure for three fully solvable quantum mechanical systems, namely, the rigid rotator, the three-dimensional harmonic oscillator, and the hydrogen atom. It turns out that many features of the supermultiplet structure are the same for all three systems. These features, which we outline in general below, illustrating them explicitly for the rigid-rotator case, seem likely to be common to many quantum mechanical systems, and may be valid for elementary-particle systems also.

We begin with the case of the rigid rotator. The Hamiltonian of the system is invariant under rotations and hence the energy levels fall into multiplets, each of which furnishes a unitary irreducible representation (UIR) of the rotation group O(3). But there is a further regularity, namely, that every (one-valued, irreducible) UIR of O(3) occurs in the spectrum once and only once. The question is this: is it possible to associate this regularity with a supermultiplet furnishing the UIR of a (larger, "approximate") symmetry group? The answer is, "Yes." For any integer ν, the first n energy levels of the rotator, which furnish, re-

spectively, the 1-, 3-, 5-, $\cdots$, $(2n+1)$-dimensional UIR's of O(3), can be identified with a UIR of the group O(4) of (real) orthogonal transformations in Euclidean four-space. But that is not all—it turns out that, in addition, the infinite number of levels left over may be identified with an infinite-dimensional UIR of the group O(3,1) of (real) orthogonal transformations in Minkowski four-space. Furthermore, the partition between the O(4) and O(3,1) multiplets (i.e., the choice of ν) is arbitrary. Note that there is no question of either O(4) or O(3,1) being an invariance group of the Hamiltonian. We may refer to O(3) as the invariance group and O(4) or O(3,1) as the noninvariance group.

Let us now give a more quantitative version of these remarks. As is well known, the rigid rotator has energy levels such that each energy level corresponds to an integral value l of the total angular momentum operator. In the energy spectrum each integral value of $l = 0$, 1, 2, $\cdots$ occurs once and only once, and the $2l+1$ states corresponding to it furnish a UIR of the invariance group O(3). The first ν multiplets of O(3) constitute the symmetric tensor representation of rank ν of O(4). If we denote the generators of O(3) by $J_{\pm} = J_1 \pm iJ_2$, J_3, and the additional generators of O(4) by $K_{\pm} = K_1 \pm iK_2$, K_3, then the commutation relations of the J, K imply the slection rules

$$\Delta l = 0, \text{ for } J, \tag{1}$$

$$\Delta l = 0, \pm 1, \text{ for } K. \tag{2}$$

Hence the only nontrivial part of determining the UIR consists in identifying the reduced matrix elements $\langle l \| K \| l \rangle$, $\langle l \| K \| l-1 \rangle$, and $\langle l-1 \| K \| l \rangle = -\langle l \| K \| l-1 \rangle (2l-1)^{1/2}/(2l+1)^{1/2}$, where, for example, $\langle l \| K \| l-1 \rangle$ is defined by the relation

$$\langle lm | K_{m^1} | l-1, m-m^1 \rangle$$

$$= C^{1\ l-1\ l}_{m^1\ m-m^1\ m} \langle l \| K \| l-1 \rangle. \tag{3}$$

For the symmetric tensor representation it

turns out that $\langle l \| K \| l \rangle$ is zero, and

$$\langle l \| K \| l-1 \rangle = [l/(2l+1)]^{1/2}[(\lambda+1)^2-l^2]^{1/2},$$

$$0 \leqslant l \leqslant \lambda, \qquad (4)$$

where λ is a non-negative integer. If we choose $\lambda = \nu$ we have the O(4) UIR containing the first ν levels of the rotator m. We could also realize an infinite-dimensional UIR of the noncompact group which incorporates all the levels of the rotator except the first ν multiplets of O(3). This representation is obtained by the transcription

$$\langle l \| K' \| l-1 \rangle \rightarrow \langle l \| iK \| l-1 \rangle$$

$$= [l/(2l+1)]^{1/2}[l^2-(\lambda+1)^2]^{1/2}, \qquad (5)$$

provided we choose $\lambda = \nu$ and $l \geqslant \nu+1$.

It is interesting to note that beside the above scheme, we have the more general possibility of organizing all the levels of the rotator between the μth level and the λth level, where μ and λ are arbitrary, into one UIR of O(4). This representation is characterized by the reduced matrix elements

$$\langle l \| K \| l \rangle = [(l+1)(2l-1)]^{1/2}\mu(\lambda+1), \qquad (6)$$

$$\langle l \| K \| l-1 \rangle = [(2l+1)(l^2-\mu^2)/l]^{1/2}[(\lambda+1)^2-l^2]^{1/2}. \qquad (7)$$

If for this representation we make the transcription

$$\langle l \| K' \| l \rangle \rightarrow \langle l \| iK \| l \rangle = [(2l-1)(l+1)]^{1/2}\mu\sigma, \qquad (8)$$

$$\langle l \| K' \| l-1 \rangle \rightarrow [(2l+1)(l^2-\mu^2)/l]^{1/2}(l^2+\sigma^2)^{1/2} \qquad (9)$$

and $\lambda \rightarrow -1-i\sigma$, σ real, and $l \geqslant \mu$, then we obtain a UIR of O(3, 1). This representation contains all the levels of the rotator except those below μ.

Besides the possibility just mentioned, we have, for the rotator, still another possibility. This is to choose, instead of the first ν levels, the first ν even-numbered (or odd-numbered) levels and put then into a UIR of a "noninvariance" group. The noninvariance group in this case is SU(3). This group has five generators in addition to the generators of the subgroup O(3), and they transform with respect to O(3) as a tensor Q of rank 2. This implies the selection rules for Q

$$\Delta l = 0, \pm 1, \pm 2. \qquad (10)$$

Hence the essential part of the explicit determination of the UIR is the computation of the reduced matrix elements $\langle l \| Q \| l \rangle$, $\langle l \| Q \| l-1 \rangle$,

and $\langle l \| Q \| l-2 \rangle$. For the symmetric tensor representation of SU(3) these turn out to be

$$\langle l \| Q \| l \rangle = -(2\lambda+3)\left[\frac{l(l+1)(2l+1)}{3(2l-1)(2l+3)}\right]^{1/2},$$

$$\langle l \| Q \| l-2 \rangle$$

$$= [\lambda(\lambda+3)-(l+1)(l-2)]^{1/2}[2l(l-1)/(2l-1)]^{1/2} \qquad (11)$$

with $\langle l \| Q \| l-1 \rangle$ equal to zero.

If we choose $\lambda = 2\nu$, we obtain a UIR of SU(3) which contains the first ν even-numbered levels of the rotator. If we choose $\lambda = 2\nu+1$, we obtain the first ν odd-numbered levels. Furthermore, if we make the formal transcription

$$\langle l \| Q' \| l \rangle \rightarrow \langle l \| iQ \| l \rangle = 2\mu\left[\frac{l(l+1)(2l+1)}{3(2l-1)(2l+3)}\right]^{1/2}, \qquad (12)$$

$$\langle l \| Q' \| l-2 \rangle \rightarrow \langle l \| iQ \| l \rangle = [(l+1)(l-2)$$

$$+ \mu^2+9/4]^{1/2}[2l(l-1)/(2l-1)]^{1/2}, \qquad (13)$$

where $\lambda \rightarrow -\frac{3}{2}+i\mu$, μ real, we obtain a UIR of the noncompact group SL(3, R). According to whether we take the lowest l value equal to zero or one, we obtain all the even-numbered or all the odd-numbered energy levels. Note, however, that in this (double-jump) case there is no possibility of obtaining all the even (or odd) levels except the first ν, for arbitrary ν. Only $\nu = 0$ or 1 are possible.

In both these methods of organizing the multiplets of O(3) into supermultiplets of O(4) and O(3, 1) on the one hand, and of SU(3) and SL(3, R) on the other, we see that the analytic structure of the generators (or, more precisely, of their matrix elements) of the compact group and the corresponding noncompact group are simply related.

We close our discussion of the rigid rotator by mentioning that, although the above considerations are confined to the rotator in three dimensions, the results generalize at once to the case of the rigid rotator in n dimensions, the invariance group being O(n), the noninvariance groups being O($n+1$) and O(n, 1), and the double-jump noninvariance groups being SU(n) and SL(n, R), in the general case.

Having discussed the case of the rigid rotator in some detail, we summarize briefly the general features which are encountered.
(1) There exists an invariance group g of the Hamiltonian. (2) g can be considered as a subgroup of a larger group G with a compact form G_C and a noncompact form G_{NC}, such that

the states of the first ν energy levels furnish a UIR of G_C and the states of the remaining levels a UIR of G_{NC} with ν arbitrary. (3) g can also be coundered as a subgroup of a larger double-jump group G^1 with a compact form G_C^1 and a noncompact form G_{NC}^1, such that the first ν even-numbered (or odd-numbered) levels furnish a UIR of G_C^1 and all the even levels (or all the odd levels) furnish a UIR of G_{NC}^1. (4) These properties are not confined to the three-dimensional case, but can be generalized to n dimensions in an obvious way. (5) Although the additional generators needed to form G and G^1 are not conserved quantities, they are dynamical variables, i.e., their explicit expressions in terms of the primitive dynamical variables can be written down.

A study of the harmonic oscillator and hydrogen atom (to be published elsewhere) reveals that the above features occur in these cases also. More specifically, we can deduce the results shown in Table I.

By the hydrogen atom in n dimensions is meant a system with a potential r^{-1}, where

$$r^2 = \sum_{i=1}^{n} x_i^2. \qquad (14)$$

Note that for $E > 0$, G_C has been generalized to be a noncompact group, since even the invariance group is noncompact in this case. The double-jump groups for $E > 0$ have not been calculated.

We make some comments on the results implied by the table. First we notice that the general features are the same for all three systems, although the rigid rotator has no higher symmetry (i.e., its invariance group is no larger than the obvious geometrical rotation group) while the other two systems have such a symmetry. Thus while the existence or nonexistence of a higher symmetry is of paramount interest for other purposes,[5] it does not seem to be relevant to the problem of noninvariance groups.

A second point we notice is the following role which is played by the double-jump noninvariance group. The ordinary noninvariance (single-jump) group, although it can accommodate all the energy levels, does not tell us which ones actually occur. This is on account of the arbitrariness of the lowest level which occurs in the UIR of the noncompact form (arbitrariness in ν above). The double-jump group has no such arbitrariness since its representation contains all the even (or all the odd) levels. Hence the double-jump group tells us which levels actually do occur. Whether or not this result is connected with the theory of Regge trajectories remains to be seen.

The third point we consider is perhaps the most striking, namely, the fact that for the noninvariance group, the compactness or noncompactness is not determined. On the contrary, it is at our disposal, and according to whether we are interested in a finite number of levels or an infinite number (all the levels, or all but a finite number), we may choose the relevant compact or noncompact form and put the levels into an appropriate UIR of that form.

We conclude by considering the question of the uniqueness of the noninvariance groups. The fact that we can include not only all the energy levels in a UIR of G_{NC} but also all the levels up to an arbitrary one in a UIR of G_C provides us with a sense in which, in our three cases, the noninvariance groups are unique. The uniqueness is expressed formally by the following theorems:

Theorem A. — Let g be a classical group, with symmetric tensor representations $t(r)$, $r = 0, 1, 2, \cdots$. We look for a classical group G_C such that $G_C \supset g$ and for every integer s, there exists a UIR $T(s)$ of G_C which decomposes with respect to g according to

$$T(s) = \sum_{r=0}^{s} \oplus\, t(r). \qquad (15)$$

Table I. Invariance and noninvariance groups for dynamical systems in n dimensions.

System	Invariance group g	Noninvariance group		Noninvariance double-jump group	
--------	------	G_C	G_{NC}	G_C^1	G_{NC}^1
Rigid rotator	$O(n)$	$O(n+1)$	$O(n,1)$	$SU(n)$	$SL(n,R)$
Harmonic oscillator	$SU(n)$	$SU(n+1)$	$SU(n,1)$	$\cdots$	$\cdots$
Hydrogen atom, $E < 0$.	$O(n+1)$	$O(n+2)$	$O(n+1,1)$	$SU(n+1)$	$SL(n+1,R)$
Hydrogen atom, $E > 0$.	$O(n,1)$	$O(n+1,1)$	$O(n,2)$		

Then,

$$\text{if } g = SU(n), \quad G_C = SU(n+1) \text{ uniquely,}$$

$$\text{if } g = O(n), \quad G_C + O(n+1) \text{ uniquely,}$$

$$\text{if } g = Sp(2n), \quad G_C = SU(2n+1) \text{ uniquely.}$$

In this sense the single-jump noninvariance groups of our table are unique.

Theorem B.—With g and $t(r)$ as in theorem A, we look for a $G_C \supset g$ and a $T(s)$ which decomposes according to

$$T(s) = \sum_{r=0}^{s} \oplus t(2r+x); \quad x = 0, 1. \tag{16}$$

Then,

$$\text{if } g = SU(n), \quad G_C \text{ does not exist;}$$

$$\text{if } g = O(n), \quad G_C = SU(n) \text{ uniquely;}$$

$$\text{if } g = Sp(2n), \quad G_C \text{ does not exist.}$$

In this sense the double-jump noninvariance groups are unique.

Finally, it might be asked whether the jumping stops at double jumps. That it does is shown by the following theorem:

Theorem C.—With g and $t(r)$ as in theorems A and B, there exists no classical group $G_C \supset g$ such that G_C has an irreducible representation $T(s)$ which decomposes with respect to g according to

$$T(s) = \sum_{r=0}^{s} \oplus t(pr+x); \quad 0 \leq x \leq p-1; p \geq 3. \tag{17}$$

The proofs of theorems A, B, and C are straightforward and are omitted in the interest of brevity.

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[2]For earlier work connecting broken symmetry with dynamics see, for example, R. E. Cutkosky and P. Tarjanne, Phys. Rev. 132, 1353 (1963); S. L. Glashow, Phys. Rev. 130, 2132 (1963).

[3]H. Bacry, CERN Report No. Th. 579, 1965 (to be published).

[4]See also A. O. Barut and A. Böhm, Phys. Rev. 139, B1107 (1965); Y. Dothan and Y. Ne'eman, in Proceedings of the Second Topical Conference on Resonant Particles, Ohio University, Athens, Ohio, 10-12 June 1965 (to be published).

[5]A general study of this interesting question has been initiated by J. Fris, V. Mandrosov, Ya. A. Smorodinsky, M. Uhlir, and P. Winternitz, Phys. Letters 16, 354 (1965).