intrinsic symmetry breaking) to $K_{(\underline{1},\underline{1})}$. It is therefore an attractive possibility that the S matrix itself belongs to $\mathfrak{D}_{\underline{1}} \oplus \mathfrak{D}_{\underline{1}}'$.

The conclusion of our discussion is that incorporating $M(12)$ symmetry-breaking kinetic-energy effects to all orders restores the unitarity of the S matrix but reduces the symmetry to (Poincaré group)$\otimes$SU(3). However, treating kinetic-energy effects in a perturbation-theoretic way (as done above via the kineton technique), many of the results of "exact $M12$" survive in the lowest orders in kineton emission.

I wish to thank Professor R. Blankenbecler, Professor L. F. Cook, Professor Y. Nambu, and Professor S. B. Treiman for interesting discussions on the kineton technique. Illuminating conversations on the theory of infinite-dimensional unitary representations of noncompact groups with Professor R. Blattner and Professor Harish-Chandra are gratefully acknowledged.

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[4]We do not count redundant components, and therefore the supermultiplicity (e.g., 36) is smaller than the $M(12)$ dimensionality.

[5]P. G. O. Freund, to be published.

[6]a is an arbitrary parameter.

[7]F. Gürsey and L. A. Radicati, Phys. Rev. Letters 13, 173 (1964).

[8]K. Bardakci, J. M. Cornwall, P. G. O. Freund, and B. W. Lee, Phys. Rev. Letters 14, 264 (1965). In reference 1 it was already emphasized that kineton pair emission is the phenomenon underlying the SU(6) mass formula. It is also remarkable that we predict the ω-φ mixing angle as opposed to the less powerful nonrelativistic considerations of M. Bég and V. Singh, Phys. Rev. Letters 13, 418 (1964).

[9]To lowest order, kineton emission from a vertex amounts to the emission of the spurion γK, K being any of the external momenta of the vertex. In higher order, also terms like $(\gamma_5)_{ab}(\gamma_5)_{a'b'}$ or $(\gamma_5\gamma K)_{ab}(\gamma_5\gamma K')_{a'b'}$, etc., appear.

[10]We call a particle belonging to $\underline{1}$ of $M(12)$ a superscalar.

[11]We have investigated this case in collaboration with R. Blankenbecler and L. F. Cook.

[12]Harish-Chandra, "Discrete Series for Semisimple Lie Groups I" (to be published).

[13]R. Blattner, private communication.

[14]Note that even the trivial representation has two correspondents in the discrete series of $SM(12)$.

[15]The problems of writing wave equations for the representations of the discrete series and of calculating the Clebsch-Gordan coefficients for products of idur's are not yet solved.

ANALYSIS OF A_1 PEAK IN THE $\pi\rho$ SYSTEM

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There has recently been much discussion over the theoretical interpretation of some of the peaks in meson systems which occur at the same energies as was predicted by Pais and Nauenberg[1] using the so-called Peierls mechanism.[2] A peak in the $\pi\rho$ system at energy 1.08 BeV, called the A_1 resonance, has been found by several experimental groups,[3-5] the energy being the same as the Pais-Nauenberg prediction.[6] The theoretical understanding of this peak, however, remains unsatisfactory to this day because of the argument by Goebel and others[7] that the singularity in the $\pi\rho$ production matrix element due to the $\pi\rho$ intermediate state is actually far away from the physical sheet.

In this note we wish to re-examine the reaction

$$\pi^+ + p \to \pi^+ + \pi^+ + \pi^- + p.$$

While we accept the Goebel argument, we find nevertheless that a peaking occurs at the energy originally suggested. This is due to specific kinematic features of the $\pi^+ p$ reaction. We find these features:

(1) The interference between the two modes of $\pi^+\rho^0$ production, viz. $\pi_a^+(\pi^-\pi_b^+)$ vs $(\pi_a^+\pi^-)$

$\times\pi_b^+$, leads to a peak at the Pais-Nauenberg energy. This peak would, however, almost be swamped by the background were it not for the N^* subtraction.

(2) Experimentally, the predominance of $N^*\rho$ production necessitates the subtraction of $N_a^* = (\pi_a^+ p)$ and $N_b^* = (\pi_b^+ p)$ "backgrounds." But a substantial part of the $\pi_a^+(\pi^-\pi_b^+) = \pi_a^+\rho_{0b}$ phase space is automatically suppressed by the N_b^* subtraction, and the subsequent effect on the expected theoretical rate of $\pi\rho$ production is quite marked.

We discuss in detail the reaction

$$\pi^+ + p \to \pi^+ + \pi^+ + \pi^- + p.$$

We define (see Fig. 1)

$$\sigma \equiv -(p_1 + k)^2, \quad \tau \equiv -(p_2 + k)^2,$$

$$\nu_1 \equiv -(p + p_1)^2, \quad \nu_2 \equiv -(p + p_2)^2,$$

$$t \equiv -(p_1 + p_2 + k)^2, \quad s \equiv -(p + p_1 + p_2 + k)^2,$$

$$P_\mu = (p + p_1 + p_2 + k)_\mu = \text{total initial momentum},$$

$$Q_\mu = (p_1 + p_2 + k)_\mu = \text{total } (\pi^+ + \pi^+ + \pi^-) \text{ momentum}.$$

We shall also call $\mu = \text{mass }(\pi)$; $m = \text{mass }(p)$,

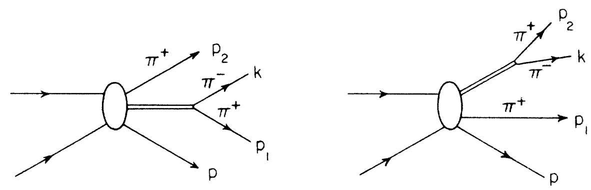

FIG. 1. Bose symmetrization of the $\pi\rho$ final state.

$M = \text{mass }(\rho)$, and $\Gamma = 100$ MeV ρ width.

We now show that the interference between the two modes of $\pi^+\rho^0$ production gives a peak at the Peierls energy. Let us write the total matrix element symbolically as

$$M = \frac{\mathfrak{M}_a\sqrt{\Gamma}}{\sigma - M^2 + iM\Gamma} + \frac{\mathfrak{M}_b\sqrt{\Gamma}}{\tau - M^2 + iM\Gamma}, \tag{1}$$

for the simple illustrative case of a s-wave ρ resonance, where we have used[7] the complex mass for the ρ meson. The ρ-meson singularity of M is close to the physical sheet, whereas the Peierls singularity of $\mathfrak{M}$ may not, according to the Goebel argument, be close to the physical region.

The total rate for $\pi\rho$ production is (over all phase space)

$$R = \int dt \int \frac{d^3p\, d^3Q}{4p_0 Q_0}\, \delta(p + Q - P) \int \frac{d^3p_1 d^3p_2 d^3k}{8p_{10}p_{20}k_0}\, \delta(Q - p_1 - p_2 - k)\, |M|^2.$$

Accepting the Goebel result, we assume $\mathfrak{M}_a, \mathfrak{M}_b$ to be relatively constant over the phase-space region of interest, in which case Bose statistics demands $\mathfrak{M}_a = \mathfrak{M}_b$. For our purpose here, we ignore the presence of a genuine resonance at $t = (1.25\ \text{BeV})^2$.

The expression for the rate then reduces to

$$R = \int dt \frac{\pi[s-(\sqrt{t}+m)^2]^{1/2}[s-(\sqrt{t}-m)^2]^{1/2}}{2s} |\mathfrak{M}|^2 \int d\sigma \int_{\tau_0}^{\tau_1} d\tau \frac{\pi^2}{4t} \left\{ \frac{\Gamma}{(\sigma-M^2)^2 + M^2\Gamma^2} + \frac{\Gamma}{(\tau-M^2)^2+M^2\Gamma^2} + 2\Gamma\,\mathrm{Re}\left(\frac{1}{\sigma-M^2-iM\Gamma}\,\frac{1}{\tau-M^2+iM\Gamma}\right)\right\}. \tag{2}$$

The two squared terms give a smoothly varying function of the type $(\Gamma/M \to 0)$

$$\frac{|\mathfrak{M}|^2}{M}\left\{\frac{\pi[s-(\sqrt{t}+m)^2]^{1/2}[s-(\sqrt{t}-m)^2]^{1/2}}{2s}\right\}\frac{\pi^2}{4t}\left\{\pi[t-(M+\mu)^2]^{1/2}[t-(M-\mu)^2]^{1/2}\left(1-\frac{4\mu^2}{M^2}\right)^{1/2}\right\}\times 2. \tag{3}$$

The interference term is appreciable only for that range of t where both $\sigma = M^2$ and also $\tau = M^2$. This range of t can readily be worked out to be

$$2M^2 + \mu^2 \leq t \leq (M^2 - \mu^2)^2/\mu^2.$$

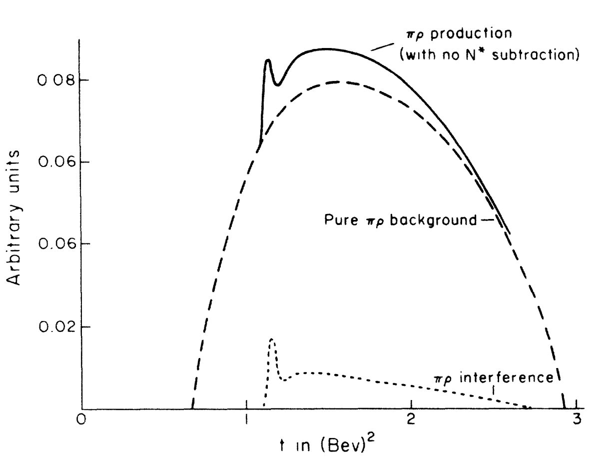

FIG. 2. Rate for $\pi\rho$ production with no N^* subtraction (plotted with $|\mathfrak{M}|^2$ assumed constant). $s=(\text{total energy})^2$ is 7.023 $(\text{BeV})^2$ corresponding to a 3.3 BeV/c incident beam.

The interference term integrates numerically to a curve as shown in Fig. 2. The peak will very easily be swamped by the pure $\pi\rho$ background.

The reason for the interference peak to be in the same energy region as the Peierls singularity peak is almost trivial. The Peierls singularity occurs over that range of t where the $\pi_a^+\rho_b^0$ can become a $\pi_b^+\rho_a^0$ state by reshuffling one real, physical pion (Fig. 3). But that is the same range of energy where $(\pi_a^+\pi^-)$ and $(\pi^-\pi_b^+)$ can both simultaneously have the ρ meson mass, which is the interference region.

We proceed now to discuss the effect due to

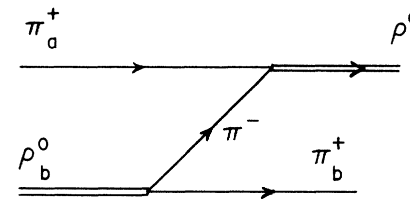

FIG. 3. Peierls mechanism for $\pi^+\rho^0$.

subtraction of N^* background. Experimentally, it was found that $\pi^+ + p \to \rho^0 + N^{*++}$ dominates the observed reaction $\pi^+ + p \to \pi^+ + \pi^+ + \pi^- + p$, and it was therefore necessary to subtract out the so-called N^{*++} background.

The subtraction procedure consists of cutting out that part of the phase space where (π_a^+p) and/or (π_b^+p) form a N^{*++} mass. In other words, the rate of $\pi^+\pi^+\pi^-p$ production that is to be compared with experiment is

$$R_{\text{exp}} = \int d[\mathcal{P} - \mathcal{P}(N_a^*) - \mathcal{P}(N_b^*) + \mathcal{P}(N_{ab}^*)]\,|M|^2, \quad (4)$$

where $\mathcal{P}(N_a^*)$ is the N^* phase space due to (π_a^+p) combination, similarly with $\mathcal{P}(N_b^*)$, and $\mathcal{P}(N_{ab}^*)$ is the overlap phase-space region where both (π_a^+p) and (π_b^+p) fall in the N^* band.

Note that with this subtraction procedure, the $N^*\rho$ part of the total matrix element is completely suppressed, and it does not matter if $\mathfrak{M}_a, \mathfrak{M}_b$ have induced variations in the N^* region.[8]

To illustrate quantitatively the effect of such a subtraction procedure on $\pi\rho$ production, we carry out a simplified analysis of Eq. (4). For the integration over the N_a^*, N_b^*, N_{ab}^* region, a good approximation for $|M|^2$ is

$$|M|^2 \cong \frac{|\mathfrak{M}|^2}{M}\{\pi\delta(\sigma-M^2) + \pi\delta(\tau-M^2) + \pi^2\Gamma M\,\delta(\sigma-M^2)\delta(\tau-M^2)\}, \quad (5)$$

where the first two terms are the background, and the third term represents the interference. The background contribution to the $N_a^* + N_b^*$ integral is

$$\int dt\int d\nu_1\left\{\frac{\pi^4}{4s}\left(1-\frac{4\mu^2}{M^2}\right)^{1/2}\right\}$$
$$+\int dt\int d\nu_1\left\{\frac{\pi^4}{4s}\ln\frac{(s-m^2-t)(\nu_1-m^2-\mu^2)+[(s-m^2-t)^2-4m^2t]^{1/2}[(\nu_1-m^2-\mu^2)^2-4\mu^2m^2]^{1/2}}{(s-m^2-t)(\nu_1-m^2-\mu^2)-[(s-m^2-t)^2-4m^2t]^{1/2}[(\nu_1-m^2-\mu^2)^2-4\mu^2m^2]^{1/2}}\right\}, \quad (6)$$

where the limits of integration for the first term are those appropriate to the phase-space configuration

$$(pp_1) = N^*, \quad (p_2k) = \rho,$$

while the second term has limits corresponding to

$$(pp_1) = N^*, \quad (p_1k) = \rho,$$

and ν_1 is integrated over $(m^*-\Delta)^2$ to $(m^*+\Delta)^2$ wherever permitted. (n^* = mass of N^* = 1.238

BeV, $\Delta = 0.100$ BeV.)

These subtraction terms are not small compared with the pure $\pi\rho$ background. The reason is kinematical. For example, fix $\nu_1 = -(p+p_1)^2 = (1.238)^2$, then in the $\pi\rho$ rest frame ($\vec{Q}=0$), it is easy to show that when p_1 forms a ρ meson with k, it also at the same time falls in the ν_1 region when combined with p. The range of values of t in the $\pi\rho$ spectrum for which this is true varies, of course, according to the chosen value for ν_1.

The overlap term $d\mathcal{O}_{N_{ab}^*}$ can also be integrated to yield

$$\int dt \int d\nu_1 \int d\nu_2 \frac{\pi^4}{4s} [(s+\mu^2-\nu_2-t)^2-4m^2M^2]^{-1/2}, \tag{7}$$

and the interference term contribution to the $N_a^* + N_b^*$ integral is

$$\int dt \int d\nu_1 \frac{\pi^4}{4s} (2\pi\Gamma M)[t-(M+\mu)^2]^{-1/2}[t-(M-\mu)^2]^{-1/2}, \tag{8}$$

omitting the interference contribution to N_{ab}^*, which is negligible.

The point now to note is that in the neighborhood of $t=1$ the N_a^* and N_b^* subtraction serves to suppress quite effectively the pure $\pi\rho$ background, and the interference $\pi\rho$ peak therefore survives [Fig. 4(a)]. The remainder of the $\pi\rho$ spectrum ($t \gtrsim 1.5$) is not as much suppressed.

An interesting point now arises with the observed reaction

$$\pi^+ + p \to \pi^+ + \pi^0 + \pi^0 + p.$$

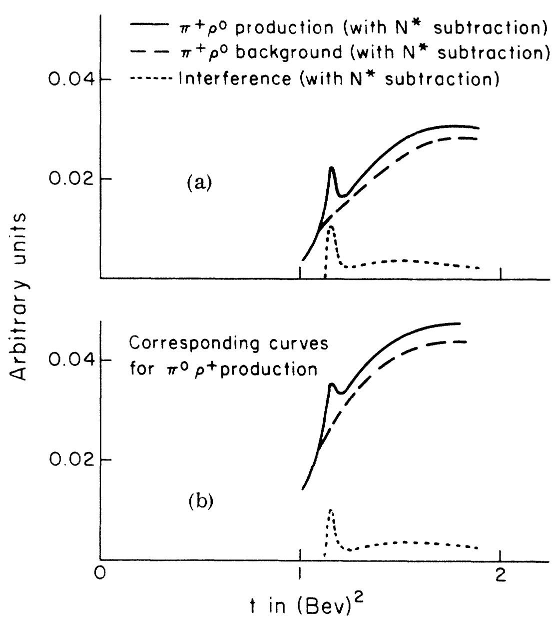

FIG. 4. (a) Rate for $\pi^+\rho^0$ production with (π_1^+p), (π_2^+p) $=N^*$ band subtracted out ($|\mathfrak{M}|^2$ assumed constant). s is still at 7.023 (BeV)^2. Note that this graph is plotted on the same scale as Fig. 2. (b) Corresponding rate for $\pi^0\rho^+$ production in the same 3.3-BeV/c experiment.

Here only one N^* combination is subtracted out, viz. (π^+p). In our present notation it means that only the $(pp_1)=N^*$, $(p_1k)=\rho$ configuration is subtracted, i.e., Eq. (6). This one subtraction, however, is not sufficient to prevent the rapidly rising pure $\pi\rho$ background from almost swamping the interference peak. This would explain the experimental fact that no sizable 1.09-BeV $\pi\rho$ resonance seems to be present in the $\pi^0\rho^+$ channel.

The interference nature of the peak produced here agrees with the observed data (see, e.g., reference 3) in the sense that most of the events contributing to the A_1 peak consist of the so-called "double-ρ" events, i.e., events in that region of the $d\sigma d\tau$ Dalitz plot where the two ρ bands cross.

In this paper we have not analyzed our model in terms of angular distributions (i.e., the $d\sigma d\tau$ Dalitz plot). A realistic calculation would have to include the effects of the ρ spin, which affect the $d\sigma d\tau$ distribution. However, it is to be expected that the effects of N^* subtraction are not uniform along the ρ bands. That is to say, in the Dalitz plot, for fixed σ in the ρ band, the depletion in the population density due to N^* subtraction is not independent of τ. It is thus not correct to dismiss the "double-ρ" events as being accountable by simple overlap of two assumed uniform ρ bands. This simple test was used to reject the A_1 as an interference enhancement in reference 3.

A more direct test[9] for the interference nature for the A_1 peak may be the following. Take $\pi\rho$ events with σ, τ satisfying $M^2 \leq \sigma, \tau \leq M^2 + 2M\Gamma$, and call these $\pi\rho_>$ events. Assuming that after the N^* subtraction only the interference peak survives, it can readily be shown from the nature of the overlap integral ($\mu \to 0$)

$$\frac{\pi^2 \Gamma M}{2t} \int_{\pi\rho_> \text{ events}} d\sigma d\tau \times \text{Re}\left\{\frac{1}{\sigma - M^2 + iM\Gamma} \cdot \frac{1}{\tau - M^2 - iM\Gamma}\right\} \qquad (9)$$

that the peak shifts from $t = 2M^2$ to

$$t = 2M^2 + 4M\Gamma + O(\Gamma^2),$$

while the height reduces from $(\pi^2\Gamma M/2t_0)(\frac{1}{2}\pi^2)$ to

$$\frac{\pi^2\Gamma M}{2(t_0 + 4M\Gamma)}(1.87).$$

Similarly, take $\pi\rho$ events with σ, τ satisfying $M^2 - 2M\Gamma \leq \sigma, \tau \leq M^2$; then the peak remains at $t \cong 2M^2$, while the height is now at $(\pi^2\Gamma M/2t_0) \times (1.87)$.

Experimentally, this would imply that the sample of $\pi\rho_>$ events merge with the A_2 peak, while the sample of $\pi\rho_<$ events would stay put, albeit reduced in size. In contrast, a bona fide resonance would show no shift in the position of the peak as $\pi\rho_>$ and $\pi\rho_<$ samples are taken.

Again we emphasize that in the realistic situation, the amount of shift as well as actual reduction in size of the interference peak depends somewhat on the spin correlations as well. Our model is merely suggestive.

The analysis presented in this note is clearly indicative of what a more accurate computer program might accomplish in fitting with the $\pi\rho$ data, manual calculations being an overwhelming handicap. It is quite interesting that the interference peak may in the end be the representative of the Peierls-Pais-Nauenberg peak.

I wish to gratefully acknowledge the interest and concern of Professor Pais, who first suggested the problem to me, and who, with Professor N. Khuri and Professor M. Bég, gave me many opportunities for discussion on the Peierls singularity. I am also grateful for discussions with Dr. M. Nauenberg, Dr. C. Kacser, Dr. S. F. Tuan, Dr. S. Adler, and Dr. R. F. Peierls.

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[6]C. Goebel, Phys. Rev. Letters 13, 143 (1964). References to earlier work on this subject may be found in this paper.

[7]A more complete description in terms of

$$\frac{1}{\pi}\int dm^2 \frac{M\Gamma}{(m^2-M^2)^2 + M^2\Gamma^2} \frac{1}{\sigma - m^2 + i\epsilon}$$

yields the same results as we have here when $\Gamma \to 0$.

[8]I am indebted to Professor M. Bég for pointing this out.

[9]I wish to thank Professor G. Goldhaber for mentioning this test to me in a fruitful discussion at the New York Physical Society meeting.