ing pole at the same point due to a discontinuity in δ.

The N function also has branch points at s_v and $s_v{}^*$ on the physical sheet, so the full amplitude A_{11} is analytic at these points. If we analytically continue D given by Eq. (8) into region II, we find the exponential factor gives a simple zero at $s=s_v$, so D has a square-root zero at the position of the resonance pole. This combines with the inverse square-root singularity of N to produce a simple pole in A_{11} as required.

If l is further decreased and the forces are sufficiently strong, the resonance pole moves onto the physical sheet as a bound state. During this continuation procedure both s_v and $s_v{}^*$ move onto the real axis below threshold, but to different positions. One migrates to the position s_B corresponding to the bound state; the other moves to the point s_z which corresponds to a zero of S_{11} on the physical sheet or a pole of S_{11} in region II below threshold. Also during this continuation, a simple zero emerges from the exponential factor in Eq. (8) and moves to $s=s_B$, giving D a square-root zero at $s=s_B$. Displaying the branch-point singularities in N and D at s_B and s_z, we have

$$D(s)=f_1(s)\left[\frac{s_B-s}{s_z-s}\right]^{1/2},$$

$$N(s)=f_2(s)[(s_B-s)(s_z-s)]^{-1/2}. \qquad (9)$$

Branches of the square-root functions in Eq. (9) are selected to make N and D real below threshold in the appropriate interval. It is now possible to define new N and D functions $\tilde{N}$ and $\tilde{D}$:

$$\tilde{N}(s)=N(s)\left[\frac{s_B-s}{s_z-s}\right]^{1/2},$$

$$\tilde{D}(s)=D(s)\left[\frac{s_B-s}{s_z-s}\right]^{1/2}. \qquad (10)$$

The functions $\tilde{N}$ and $\tilde{D}$ are free of extraneous branch cuts; there is now an ordinary-type CDD pole at $s=s_z$, and D has a simple zero at $s=s_B$. It is also possible to eliminate the CDD cuts even before the resonance pole has emerged onto the physical sheet as a bound state by multiplying the N and D functions by appropriate factors. The resulting N and D functions, however, are not the analytic continuations from high l. Moreover, the location of the CDD pole will generally have no simple relation to the poles or zeros of S_{11} as is the case in Eq. (10). This is discussed in more detail in reference 4.

We remark finally that it is clear from our discussion that a necessary (and for most cases sufficient) condition that the Frye-Warnock equations give the many-channel result for A_{11} is that there be no zeros of S_{11} on the physical sheet which retreat into the inelastic $l \to \infty$.

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KINETIC-ENERGY EFFECTS IN $M(12)$ THEORY

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$M(12)$ symmetry,[1,2] as has been previously emphasized,[1] is intrinsically broken. The kinetic-energy effects and the subsidiary conditions that project out the physical particle states break unavoidably the symmetry. The concept of "exact $M(12)$ symmetry" is therefore in conflict with such basic principles of quantum theory as the unitarity of the S matrix.[3] It is imperative to formulate the theory incorporating the intrinsic symmetry breaking from the beginning.

In this paper we wish to present an attempt in this direction.

In order to have a model, let us consider 36 mesons (M) grouped in a $\underline{143}$ representation of $M(12)$ in interaction with six quarks (Q) belonging to 12.[4] $M(12)$ requires the QM vertex to be[5]

$$\Gamma_{QM} = \bar{Q}^{\alpha}(p')M_{\alpha}{}^{\beta}(q)Q_{\beta}(p),$$

with

$$q = p'-p; \quad M(q) = (1/8^{1/2}m)(\gamma q + m)(\gamma_5 P + \gamma_\mu V^\mu),$$

P and V_μ being the familiar SU(3) pseudoscalar and vector meson matrices and Q the 12-component quark spinor.

Consider now the self-energy graphs of the mesons of Fig. 1. Knowing the QM vertex, we can calculate the meson self-energy parts which will be given by divergent integrals. Since we are, however, interested only in the spin-unitary-spin structure of the self-energy parts, we can provisionally cut off the integrals and thus obtain typically terms of the form

$$\mathcal{S} = \langle\langle \bar{M}M \rangle\rangle = \langle P^2 + V^2 \rangle,$$

$$\mathcal{P} = \langle\langle \bar{M}\gamma_5 M\gamma_5 \rangle\rangle = 0,$$

$$\mathcal{T} = \tfrac{1}{2}\langle\langle \sigma_{\mu\nu}\bar{M}\sigma_{\mu\nu}M \rangle\rangle = \langle P^2 - V^2 \rangle,$$

$$\mathcal{V} = \langle\langle \gamma_\mu \bar{M}\gamma_\mu M \rangle\rangle = -\langle P^2 + V^2 \rangle,$$

$$\mathcal{A} = \langle\langle i\gamma_\mu\gamma_5 \bar{M} i\gamma_\mu\gamma_5 M \rangle\rangle = \langle V^2 - 3P^2 \rangle$$

[with $\bar{M}(q) = M(-q)$] multiplying the integrals. In (3) we have used the notations

$$\langle\langle A \rangle\rangle = \operatorname*{Tr}_{\text{Dirac}} \operatorname*{Tr}_{\text{SU(3)}} A \text{ and } \langle A \rangle = \operatorname*{Tr}_{\text{SU(3)}} A.$$

Whereas $\mathcal{S}$, $\mathcal{P}$, and $\mathcal{V}$ contribute equally to the P and V masses, this is not the case for $\mathcal{T}$ and $\mathcal{A}$. We thus see that via the $\mathcal{A}$ and $\mathcal{T}$ terms the degeneracy of the P and V mesons is lifted. Inserting a Gell-Mann–Okubo term in the quark propagator, we then get, to lowest order in SU(3) breaking, terms of the type

$$\langle\langle \bar{M}\{\lambda_8, \bar{M}\}\rangle\rangle \text{ and } \langle\langle \bar{M}\gamma_\mu\{M, \lambda_8\}\gamma_\mu \rangle\rangle,$$

which will induce splittings in the SU(3) multiplets. We obtain in this way the well-known mass formulas

$$\varphi - k^* = k^* - \rho, \quad k - \pi = k^* - \rho, \quad \omega = \rho.$$

In order to avoid $X(960)-\eta(550)$ mixing, we have to include also terms of the form

$$\sum_{i=0}^{8} \langle P\lambda_i P\lambda_i \rangle.$$

Such terms can be constructed in various ways, e.g.,

$$\sum_{i=0}^{8} \langle\langle a(\bar{M}\gamma_\mu\lambda_i M\gamma_\mu\lambda_i + \bar{M}i\gamma_\mu\lambda_i Mi\gamma_\mu\lambda i)$$
$$+ (1-a)(\bar{M}\lambda_i M\lambda_i + \bar{M}\gamma_5\lambda_i M\gamma_5\lambda_i$$
$$+ \tfrac{1}{2}M\sigma_{\mu\nu}\lambda_i M\sigma_{\mu\nu}\lambda_i)\rangle\rangle,$$

and[6] they appear indeed from graphs of the type of Fig. 1. For baryons the same procedure yields the well-known Gürsey-Radicati mass formula.[7]

What we have done so far is a systematic application of the technique of "kineton emission," discussed in an earlier publication,[8] to the mass splittings of the mesons.

We now wish to apply the kineton technique to vertex functions and scattering amplitudes. Consider the meson-baryon vertex function Γ_{MB}. If we insert the corrections to all orders in kineton emission, i.e., consider all graphs of the type of Fig. 2, the symmetry is

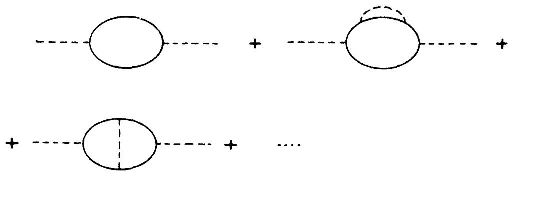

FIG. 1. Meson self-energy graphs.

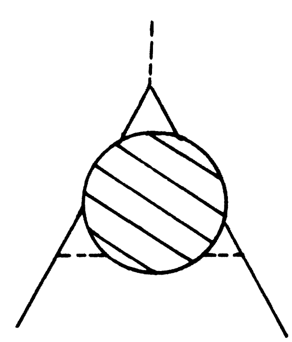

FIG. 2. Corrections to $M(12)$ vertex.

broken down to (Poincaré group)$\otimes$SU(3). Limiting ourselves to lowest order in kineton emission,[9] and making extensive use of the Bargmann-Wigner equation, we find

$$\Gamma_{MB}=F_1(q^2)\bar{\psi}^{\alpha\beta\gamma}(p')\psi_{\alpha\beta\delta}(p)M_\gamma{}^\delta(q)$$
$$+F_2(q^2)\bar{\psi}^{\alpha\beta\gamma}(p')\psi_{\alpha\beta\gamma}(p)[\gamma(p+p')M(q)]_\delta{}^\delta$$

(with $q=p'-p$), and only one new form factor is introduced in addition to the "exact $M(12)$" form factor. Since the second term in (4) only couples the SU(3) singlet $[(\frac{2}{3})^{1/2}\omega-(\frac{1}{3})^{1/2}\varphi]$ vector meson to baryons, it is obvious that the predictions of $M(12)$ for electromagnetic and weak form factors and pseudoscalar meson-baryon couplings ($D/F=\frac{3}{2}$) stay unchanged.

When we apply the same technique to scattering processes, the violation of unitarity is stepwise eliminated. For the simple case of superscalar-quark (SQ) scattering, we find that to lowest order in kineton emission, the scattering amplitude is of the form

$$\bar{u}_Q(p')u_Q(p)A(s,t)+\bar{u}_Q(p')\gamma(k+k')u_Q(p)B(s,t)$$

(in an obvious notation), and thus unitarity is restored. This is not so in general, however. In particular, for the less trivial case of SM scattering, unitarity is restored only in higher orders of kineton emission,[10] whereas to lowest order unitarity is still violated.[11]

We now wish to point out a possible connection of the theory of kineton emission with the infinite-dimensional unitary representations (idur's) of $SM(12)$. In the following (without loss of generality), we restrict ourselves to the semisimple noncompact group $SM(12)$, obtained by dividing a U(1) out of $M(12)$. Let us consider a supermultiplet of particles at rest belonging to the representation D of SW(6), the maximal compact subgroup of $SM(12)$. The kinetic energy belongs to the representation $T=(\underline{6},\underline{6}^*)+(\underline{6}^*,\underline{6})$ of $SM(12)$.[1] If the particles of D are moving, they will not any more belong to the pure representation D of SW(6), but rather pick up (due to kinetic-energy effects) components from all representations contained in the reduction of

$$K_D=\sum_{n=0}^{\infty}D\otimes T^n.$$

Thus a moving supermultiplet, due to the intrinsic symmetry breaking, is to be described by an infinite sequence K_D of SW(6) representations. We shall call K_D the kinetic sequence of D. Since a finite-dimensional nonunitary representation F of $SM(12)$ contains only a finite number of SW(6) representations, it does not completely describe a moving supermultiplet, and this is presumably the cause for the "unitarity problem."[3] The natural mathematical objects to describe moving supermultiplets are thus idur's of $SM(12)$. In general, there are continuously many idur's for a noncompact group. Because of the rank of $SM(12)$ being equal to that of its maximal compact subgroup SW(6), $SM(12)$ has, in addition, a discrete series of idur's. This discrete series has been studied by Harish-Chandra,[12] who has shown that for any finite-dimensional representation F of a semisimple noncompact group G there exist precisely two idur's, $\mathfrak{D}_F$ and $\mathfrak{D}_F'$ in the discrete series of G that have the same infinitesimal character as F and can be represented over the space of holomorphic functions on G/A, A being the maximal Abelian subgroup of G. $\mathfrak{D}_F$ and $\mathfrak{D}_F'$ can be reduced with respect to SW(6) using a set of rules due to Blattner.[13] As a result of such a reduction we find[14]

$$\mathfrak{D}_1=(\underline{1},\underline{1})+(\underline{6},\underline{6}^*)+(\underline{21},\underline{21}^*)+(\underline{15},\underline{21}^*)+(\underline{6}^*,\underline{6})+\cdots,$$
$$\mathfrak{D}_{12}=(\underline{6},\underline{1})+(\underline{21},\underline{6}^*)+(\underline{15},\underline{6}^*)+\cdots,$$
$$\mathfrak{D}_{143}=(\underline{6},\underline{6}^*)+(\underline{21},\underline{15}^*)+(\underline{56},\underline{56}^*)+(\underline{70},\underline{70}^*)+\cdots,$$
$$\mathfrak{D}_{364}=(\underline{56},\underline{1})+(\underline{126},\underline{6}^*)+\cdots.$$

From the lower terms that we have worked out we see that $\mathfrak{D}_F$ contains that SW(6) representation which is obtained from F at rest and members of its kinetic sequence. The $\mathfrak{D}_F'$'s contain the parity conjugate representation and members of its kinetic sequence [e.g., $\mathfrak{D}_{364}'$ starts with $(\underline{1},\underline{56})$].

It is an interesting possibility that $\mathfrak{D}_F\oplus\mathfrak{D}_F'$ contains a parity-complete SW(6) representation, its full kinetic sequence and nothing else. In all cases considered so far we have found no exception to this rule, though we were not able to prove it for the whole infinite kinetic sequences of SW(6) representations.[15]

It is interesting to remark that the S matrix calculated from a Lagrangian with an $M(12)$-invariant interaction part will belong (due to intrinsic symmetry breaking) to $K_{(\underline{1},\underline{1})}$. It is therefore an attractive possibility that the S matrix itself belongs to $\mathfrak{D}_{\underline{1}}\oplus\mathfrak{D}_{\underline{1}}'$.

The conclusion of our discussion is that incorporating $M(12)$ symmetry-breaking kinetic-energy effects to all orders restores the unitarity of the S matrix but reduces the symmetry to (Poincaré group)$\otimes$SU(3). However, treating kinetic-energy effects in a perturbation-theoretic way (as done above via the kineton technique), many of the results of "exact $M12$" survive in the lowest orders in kineton emission.

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[4]We do not count redundant components, and therefore the supermultiplicity (e.g., 36) is smaller than the $M(12)$ dimensionality.

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[9]To lowest order, kineton emission from a vertex amounts to the emission of the spurion γK, K being any of the external momenta of the vertex. In higher order, also terms like $(\gamma_5)_{ab}(\gamma_5)_{a'b'}$ or $(\gamma_5\gamma K)_{ab}(\gamma_5\gamma K')_{a'b'}$, etc., appear.

[10]We call a particle belonging to $\underline{1}$ of $M(12)$ a superscalar.

[11]We have investigated this case in collaboration with R. Blankenbecler and L. F. Cook.

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[14]Note that even the trivial representation has two correspondents in the discrete series of $SM(12)$.

[15]The problems of writing wave equations for the representations of the discrete series and of calculating the Clebsch-Gordan coefficients for products of idur's are not yet solved.

ANALYSIS OF A_1 PEAK IN THE $\pi\rho$ SYSTEM

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There has recently been much discussion over the theoretical interpretation of some of the peaks in meson systems which occur at the same energies as was predicted by Pais and Nauenberg[1] using the so-called Peierls mechanism.[2] A peak in the $\pi\rho$ system at energy 1.08 BeV, called the A_1 resonance, has been found by several experimental groups,[3-5] the energy being the same as the Pais-Nauenberg prediction.[6] The theoretical understanding of this peak, however, remains unsatisfactory to this day because of the argument by Goebel and others[7] that the singularity in the $\pi\rho$ production matrix element due to the $\pi\rho$ intermediate state is actually far away from the physical sheet.

In this note we wish to re-examine the reaction

$$\pi^+ + p \rightarrow \pi^+ + \pi^+ + \pi^- + p.$$

While we accept the Goebel argument, we find nevertheless that a peaking occurs at the energy originally suggested. This is due to specific kinematic features of the π^+p reaction. We find these features:

(1) The interference between the two modes of $\pi^+\rho^0$ production, viz. $\pi_a^{\ +}(\pi^-\pi_b^{\ +})$ vs $(\pi_a^{\ +}\pi^-)$