MASS DIFFERENCES AND LIE ALGEBRAS OF FINITE ORDER

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Recently a large number of papers have appeared in which the possibilities of combining Lorentz invariance and internal symmetry have been discussed.[1] One possibility which presents itself is that the (inhomogeneous) Lorentz group $\mathfrak{L}$ is a subgroup of a larger symmetry group. In this connection the question arises as to whether it might be possible to explain within the context of the larger symmetry group the mass differences which are observed to occur within the multiplets of the ordinary internal symmetry groups. In a previous paper, in which attention was confined to Lie algebras rather than Lie (or other) groups, the possibilities for imbedding the Lie algebra L of $\mathfrak{L}$ in a larger Lie algebra G were investigated, and it was shown that if G were of finite order, the ways in which this could be done could be classified into four classes. It was then shown that within the context of the first two classes of Lie algebras the mass differences could not be explained.

In the present paper we wish to extend the latter result by establishing a theorem which shows that, if the mass operator is self-adjoint, and the masses of the particles are regarded as discrete points in the mass spectrum, then the mass differences cannot be explained within the context of any Lie algebra of finite order. The theorem is as follows:

Theorem. – Let L denote the Lie algebra of the inhomogeneous Lorentz group $\mathfrak{L}$, G denote the Lie algebra of any Lie group $\mathcal{G}$ of finite order r, of which L is a subgroup, and let H denote Hilbert space on which any representation of G operates. Let P_μ denote the infinitesimal generators of the space-time translations in L. If the mass operator

$$P^2 = P_\mu P^\mu, \tag{1}$$

and every finite power thereof, are self-adjoint on H, and if there exists a discrete point m^2 in the spectrum of P^2 on H, then the eigenspace H_m belonging[2] to the point m^2 is closed, and is invariant with respect to the operators representing the Lie algebra G.

Proof. – That H_m is closed follows from the fact that it is the eigenspace of the self-adjoint, and therefore closed, operator.[3] To prove that it is invariant with respect to the operators representing the Lie algebra G, it is convenient to establish first the following two Lemmas.

Lemma I. – Let E denote any element in the Lie algebra of G, and in particular let $M_{\mu\nu}$ and P_σ denote the base elements in the Lie algebra of L, so that

$$[M_{\mu\nu}, P_\sigma] = g_{\nu\sigma} P_\mu - g_{\mu\sigma} P_\nu, \tag{2}$$

$$[P_\mu, P_\nu] = 0. \tag{3}$$

Let U_n denote any commutator of the form

$$[P_1[P_2[P_3 \cdots [P_n, E] \cdots]]], \tag{4}$$

where each P_r, $r = 1, \cdots, n$, is equal to one of the four base elements P_σ. Then there exists a finite integer N, such that

$$U_n = 0 \text{ for } n \geq N. \tag{5}$$

Proof. – Let $D(E)$ denote the matrices of the adjoint representation of the Lie algebra of G. That is to say, if E_a, $a = 1, \cdots, r$, is a basis in the Lie algebra of G,

$$[E, E_a] = D_{ab}(E) E_b. \tag{6}$$

Then we can write U_n in the form

$$U_n = D(P_1) D(P_2) \cdots D(P_n) E. \tag{7}$$

Hence to establish the Lemma, it is sufficient to prove that there exists a finite integer N, such that

$$D(P_1) D(P_2) \cdots D(P_n) = 0 \text{ for } n \geq N. \tag{8}$$

To prove this we note that from (2),

$$D(P_\mu) = [D(P_\nu), D(M_{\mu\nu})], \tag{9}$$

for any $\nu \neq \mu$ (ν not summed), whence from (3),

$$D^n(P_\mu) = [D(P_\nu), D(M_{\mu\nu}) D^{n-1}(P_\mu)], \tag{10}$$

for any $\nu \neq \mu$ (ν, μ not summed) and for any integer n. But the adjoint representation of a Lie algebra of finite order is finite dimensional, and in a finite-dimensional space, the trace of any commutator is zero. Hence,

$$\mathrm{Tr} D^n(P_\mu) = 0, \tag{11}$$

for any integer n. Hence the four matrices $D(P_\sigma)$ are nilpotent. Hence for each of them there exists a finite integer n_σ such that

$$D^n(P_\sigma) = 0 \text{ for } n \geq n_\sigma. \tag{12}$$

Since the four $D(P_\sigma)$ commute, we obtain the required result (8) by choosing $N = 4\times$[maximum (n_σ)].

Corollary. – If V_n denotes the commutator

$$V_n = [P^2[P^2[P^2 \cdots [P^2, E]\cdots]]], \tag{13}$$

which contains n P^2's, then

$$V_n = 0 \text{ for } n \geq N. \tag{14}$$

Proof. – From the relation

$$\begin{aligned}[][P^2, U_r] &= g^{\lambda\sigma}\{P_\lambda[P_\sigma, U_r] + [P_\sigma, U_r]P_\lambda\} \\ &= g^{\lambda\sigma}\{2P_\lambda[P_\sigma, U_r] - [P_\lambda[P_\sigma, U_r]]\} \\ &= g^{\lambda\sigma}\{2P_\lambda U_{r+1} - U_{r+2}\},\end{aligned} \tag{15}$$

if follows easily by induction in r that

$$V_n = \sum_{r=n}^{r=2n} C_r U_r, \tag{16}$$

where the C_r are polynomials in the P's. But, from Lemma I, the right-hand side of this equation is zero for $n \geq N$, Q.E.D.

Lemma II. – Let A be a linear operator on a Hilbert space H, such that A and all finite powers of A are self-adjoint. If there exists a vector $|h\rangle$ in H such that

$$A^N|h\rangle = 0, \tag{17}$$

for some finite integer N, then

$$A|h\rangle = 0. \tag{18}$$

Proof. – Let m be any integer such that $2^m \geq N$. Then from (17),

$$A^{2^m}|h\rangle = 0. \tag{19}$$

A trivial consequence of this equation is that $A^{2^m}|h\rangle$ exists. But if $A^{2^m}|h\rangle$ exists, $A^{2^m-1}|h\rangle$ exists and lies in the domain of A^{2^m-1}. Further, if $A^{2^m-1}|h\rangle$ exists, $|h\rangle$ lies in the domain of A^{2^m-1}. Thus (19) implies that $|h\rangle$ and $A^{2^m-1}|h\rangle$ lie in the domain of A^{2^m-1} and that $|h\rangle$ lies in the domain of A^{2^m}. Hence, since A^{2^m-1} is self-adjoint,[3]

$$\langle A^{2^m-1}h, A^{2^m-1}h\rangle = \langle h, A^{2^m}h\rangle. \tag{20}$$

But from (19), the right-hand side of this equation is zero. Hence,

$$\langle A^{2^m-1}h, A^{2^m-1}h\rangle = 0. \tag{21}$$

Since the metric is positive definite, we then have

$$A^{2^m-1}|h\rangle = 0. \tag{22}$$

Comparing this equation with (19), we see that we can deduce in an exactly similar way

$$A^{2^m-2}|h\rangle, \tag{23}$$

and similarly,

$$A^{2^m-3}|h\rangle = A^{2^m-4}|h\rangle = \cdots = A^2|h\rangle = A|h\rangle = 0, \tag{24}$$

Q.E.D.

We proceed now to the proof of the main theorem. Let E be an infinitesimal generator of the transformations representing G on H. Let $|h_m\rangle$ be any vector in H_m on which E is defined, and consider the vector

$$E|h_m\rangle. \tag{25}$$

Since, by definition of H_m,

$$(P^2 - m^2)|h_m\rangle = 0, \tag{26}$$

we have

$$\begin{aligned}&(P^2 - m^2)^N E|h_m\rangle \\ &\quad = [P^2[P^2[P^2\cdots[P^2, E]\cdots]]]|h_m\rangle,\end{aligned} \tag{27}$$

where the commutator on the right contains N P^2's. But, by the corollary to Lemma I, this commutator is zero. Hence,

$$(P^2 - m^2)^N E|h_m\rangle = 0. \tag{28}$$

But since every finite power of P^2 is self-adjoint, m^2 is real, and every finite power of (P^2-m^2) is self-adjoint. Hence, from Lemma II,

$$(P^2-m^2)E|h_m\rangle = 0. \qquad (29)$$

Hence, by definition,

$$E|h_m\rangle \subset H_m. \qquad (30)$$

Thus the space H_m is invariant with respect to E.

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[1]See, for example, the references given in L. O'Raifeartaigh, Phys. Rev. Letters 14, 332 (1965), and in E. C. G. Sudarshan, Coral Gables Conference on Symmetry Principles at High Energy, edited by B. Kurşunoğlu and A. Perlmutter (W. J. Freeman & Co., San Francisco, 1965).

[2]Every discrete point in the spectrum of a self-adjoint operator is an eigenvalue of the operator, i.e., there corresponds to it a nontrivial eigenspace [N. J. Akhiezer and I. M. Glazman, Theory of Linear Operators in Hilbert Space (Frederick Ungar Publishing Company, New York, 1961), pp. 81 and 91].

[3]Akhiezer and Glazman, reference 2, pp. 80-81.

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COVARIANCE, SU(6), AND UNITARITY. M. A. B. Bég and A. Pais [Phys. Rev. Letters 14, 509 (1965)].

The discussion following Eq. (3) is geared to the following Eqs. (1) and (2):

$$T(1) = f(s,t)\bar{u}_A(\vec{p}_4)u^A(\vec{p}_1)\cdot\bar{v}_B(\vec{p}_2)v^B(\vec{p}_3), \qquad (1)$$

$$T(143) = g(s,t)\bar{u}_A(\vec{p}_4)(O_N)_B{}^A u^B(\vec{p}_1) \times \bar{v}_C(\vec{p}_2)(O_N)_D{}^C v^D(\vec{p}_3), \qquad (2)$$

not to the coupling scheme reproduced in the paper.

The closing sentence of the first paragraph on page 511 should read: "In the present case unitarity can be implemented if i is a pure singlet, though not if it is pure 143."