$$[B_{\alpha\rho\sigma}{}^{\dagger}M_{\gamma}{}^{\rho}t_{\beta}{}^{\sigma}+B_{\alpha\rho\sigma}{}^{\dagger}M_{\beta}{}^{\rho}t_{\gamma}{}^{\sigma}$$

$$+B_{\gamma\rho\sigma}{}^{\dagger}M_{\beta}{}^{\rho}t_{\alpha}{}^{\sigma}]B^{\alpha\beta\delta}x_{\delta}{}^{\gamma}. \quad (8''')$$

The first term, Eq. (8), does not contribute; the second and the third, Eqs. (8′) and (8″), give equal contributions. Eliminating the two independent parameters one finds Eqs. (1), (2), (3), and (4).

Partial inclusion of symmetry breaking, by restricting the symmetry to the subgroup SU(4)(T) ⊗SU(2)(X)⊗W(Y), leaves Eqs. (1) and (2) unchanged. Under SU(4)(T)⊗SU(2)(X) the nucleons transform as ($\underline{20}$, $\underline{1}$), Σ and Λ as ($\underline{10}$, $\underline{2}$), π as ($\underline{15}$, $\underline{1}$). The weak spurion x transforms as ($\underline{4}$, $\underline{2}$), and the angular-momentum spurion t as ($\underline{15}$, $\underline{1}$)⊕($\underline{1}$, $\underline{3}$). The main point is to observe that the ($\underline{1}$, $\underline{3}$) component of t cannot contribute because of its SU(2)(X) behavior—we call to mind the related selection rules following from conservation of G' parity.[3] The component ($\underline{15}$, $\underline{1}$) of t introduces four invariants [corresponding to the fourfold appearance of ($\underline{1}$, $\underline{1}$) in ($\underline{20}$, $\underline{1}$)⊗($\underline{15}$, $\underline{1}$)⊗($\underline{15}$, $\underline{1}$)⊗($\underline{20}$, $\underline{1}$)] in one-to-one correspondence with the analogous invariants in SU(6).

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COVARIANCE, SU(6), AND UNITARITY

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In a recent note[1] it has been shown how the method of relativistic completion of SU(6) leads to fully covariant and crossing-symmetric effective vertices and matrix elements. We continue to mean by SU(6) a group property of zero-three-momentum one-particle states. For brevity we denoted all these effective quantities as "S-matrix quantities," so that the term S matrix is used in a phenomenological sense. It was found that this completion procedure is in general not unique. The lack of uniqueness is due to the fact that there are (with the exception of the $\underline{6}$ representation) inequivalent ways in which an SU(6) representation can be boosted to momentum $\vec{p}$. The total set of ways in which this can be done is fully determined by the spin content of the SU(6) representation in hand. The inequivalent boosts are effectively indistinguishable when applied to bilinear forms (free particles), but they are effectively distinct when applied to n-point functions, $n>2$. For the meson($\underline{35}$)-baryon($\underline{56}$) three-point function, the set of covariant but inequivalent vertices was given. It was noted that the noncompact booster group $SU(12)_{\mathcal{L}}$ provides a convenient way of keeping track of the inequivalent boosts of a given SU(6) representation. In particular it was found that a unique meson-baryon vertex emerges if one assigns the $SU(12)_{\mathcal{L}}$ representations[1,2] $\underline{364}$ and $\underline{143}$ to the boosted $\underline{56}$ and $\underline{35}$, respectively. Similar results for this vertex have been obtained independently by several other authors.[2] It was further observed[1] that the same methods can be applied to any n-point function to yield covariant and crossing-symmetric answers.

Within the conventional framework of quantum mechanics and relativity theory, the description in terms of this covariant SU(6)-invariant effective S matrix is only approximate in a dynamical sense. It should indeed be recalled[1,3] that the completion procedures cannot be applied in general to a Lagrangian field theory with interaction, where the free kinetic-energy term breaks the completed SU(6) structure.[4] This inapplicability was demonstrated by the following simple counter example.[3] Take a spin-0 and a spin-1 meson, both belonging to the same 35. Compute the difference in second-order self-energies of these particles due to their SU(6)-complete coupling with 6. This difference does not vanish. However, if in the fermion loop one drops the kinetic-energy terms, then the difference does vanish. Likewise for higher order contributions.

It is clearly desirable to give an argument, more general than the naive calculation mentioned above, to establish the fact that the free kinetic energy breaks the completed SU(6). The purpose of this note is to prove this point by studying the unitarity properties of the effective S matrix. This method has an added advantage in that it is independent of whether or not a possible underlying local field theory can be treated by perturbative methods.

We also find broader conditions of validity for results previously obtained[1] with $SU(12)_\mathcal{L}$.

We first prove the following statements:

(I) It is not possible to implement unitarity for the effective S matrix in the $SU(12)_\mathcal{L}$ description. This conclusion has been reached independently by R. Blankenbecler, M. L. Goldberger, and S. B. Treiman.[5]

(II) As was observed earlier,[1] it is not necessary to require $SU(12)_\mathcal{L}$ invariance to have a covariant SU(6)-invariant S matrix. Even under such more general completion conditions unitarity still cannot be implemented.

(III) For elastic-scattering processes, unitarity can be implemented as a limit property at zero kinetic energy. This applies in particular to baryon-baryon and to baryon-meson scattering.

(IV) All these general conclusions would remain valid if one were to replace the internal-symmetry group SU(3) by any other compact group.

We prove (I) by the following counter example. Consider the scattering $S(\vec{p}_1)+\overline{S}(\vec{p}_2)\to\overline{S}(\vec{p}_3)+S(\vec{p}_4)$ of a sextet member S on an antisextet member $\overline{S}$. Under $SU(12)_\mathcal{L}$, the initial state is in $\underline{12}^*\otimes\underline{12}=\underline{1}\oplus\underline{143}$. Accordingly, the scattering amplitude $T=T(1)+T(143)$,

$$T(1)=f(s,t)\overline{u}_A(\vec{p}_1)v^A(\vec{p}_2)\cdot\overline{v}_B(\vec{p}_3)u^B(\vec{p}_4), \qquad (1)$$

$$T(143)=g(s,t)\overline{u}_A(\vec{p}_1)(O_N)_B{}^A v^B(\vec{p}_2)\times\overline{v}_C(\vec{p}_3)(O_N)_D{}^C u^D(\vec{p}_4). \qquad (2)$$

Here f,g are functions [not determined by $SU(12)_\mathcal{L}$] of the usual variables s,t; u refers to S, v to $\overline{S}$; $A,B,\cdots$ are SU(3) labels. $N=1,\cdots,143$, $(O_N)_B{}^A=(F_B{}^A,\Gamma_\lambda\delta_B{}^A,\Gamma_\lambda F_B{}^A)$. $F_B{}^A$ are the eight SU(3) generators. Γ_λ, $\lambda=1,\cdots,15$, represents the 15 generators of $SU(4)_\mathcal{L}$ which may be taken to be $(\gamma_5,\gamma_\mu,\gamma_5\gamma_\mu,\sigma_{\mu\nu})$. Thus Eq. (1) is of type S, while Eq. (2) contains P, V, A, T with equal weight. If unitarity holds, then

$$\mathcal{A}\{\langle f|T|i\rangle\}=\tfrac{1}{2}(2\pi)^4\sum_n\langle f|T^\dagger|n\rangle\langle n|T|i\rangle\delta^4(P_n-P_i); \qquad (3)$$

$\mathcal{A}$ denotes "absorptive part."

Equations (1)-(3) are sufficient to test unitarity in the elastic region. It is straightforward to verify that no linear combination of (1) and (2) is closed under unitarity. The crucial point is that the closure over the states n proceeds via

$$\sum u^{\beta B}(p)\overline{u}_{\alpha A}(p)=[(i\gamma p+m)/2m]_\alpha{}^\beta\delta_A{}^B, \qquad (4)$$

where $\sum$ is the closure sum over positive-energy states. The γp term has the following effects: When Eq. (3) is (a) applied to $T(1)$, it generates part of $T(143)$; (b) applied to $T(143)$, it breaks the balance between the equal weights for P, V, A, T. (Likewise for closure of $v\overline{v}$.)

Equation (4) illustrates several other simple points. First, imagine one asks only to implement unitarity for a fully specified SU(3) representation. This of course is trivially possible due to the compact nature of SU(3), manifest in Eq. (4) through the completeness symbol $\delta_A{}^B$. Secondly, let us forget for a moment the SU(3) aspects (which are immaterial in this context anyway) and shrink $SU(12)_\mathcal{L}$ to the 15-parameter group $SU(4)_\mathcal{L}$. Then the lack of unitarity is reduced to its essence: Neither is it possible that the scattering amplitude for fermion-antifermion scattering is purely of type S, nor of the type $P+V+A+T$, for all energies and momentum transfers, and also that we still have unitarity. This argument also proves (IV) in relation to (I).

For a compact symmetry group it is of course possible to satisfy Eq. (3) "per representation." That is, if the state i transforms according to a given irreducible representation of the group, then the issue is settled because n and f transform according to the same representation. Our example shows that in the present case not only is there no unitarity per representation, but that the same is true for the linear superposition of all allowed representations ($\underline{1}$ and $\underline{143}$ in this case).

This last point is not in evidence for the even simpler case of the scattering $S(\vec{p}_1)+\varphi(\vec{p}_2)\rightarrow S(\vec{p}_3)+\varphi(\vec{p}_4)$, where φ is a spinless meson transforming as $\underline{1}$ under $SU(12)_{\mathcal{L}}$, because there is now only one amplitude:

$$T=h(s,t)\bar{u}_A(\vec{p}_1)u^A(\vec{p}_3)\varphi(\vec{p}_2)\varphi(\vec{p}_4). \tag{5}$$

Again (I) is shown to be true, but the question of superposition does not arise.

On the other hand, Eq. (5) is useful to prove (II) by counter example. If we do not insist on $SU(12)_{\mathcal{L}}$, then in general the number of independent amplitudes increases.[1] However, if we continue to define the completion of SU(6) by (a) using boosted SU(6) representations, (b) contraction of indices according to SU(6) tensor analysis, observing the invariance under $\mathcal{L}^{\dagger}$, then still Eq. (5) remains the only possibility for constructing the scattering amplitude. This exceptional case therefore serves to prove (I) and (II) at the same time.

As a side remark, we note that both examples make use of the property Eq. (4) of spin-$\frac{1}{2}$ particles. However, for any spin >0 there are corresponding closure equations which lead to similar conclusions. Thus for spin 1, the well-known relation

$$\sum\epsilon_\mu(q)\epsilon_\nu(q)=\delta_{\mu\nu}-\frac{q_\mu q_\nu}{\mu^2} \tag{6}$$

(summing over polarization states), leads to a $q_\mu q_\nu$ term which likewise generates departures from unitarity.

Equations (4) and (6) also make evident why unitarity and $SU(12)_{\mathcal{L}}$ [or more general SU(6) completions] are compatible at zero kinetic energy. In this case Eq. (4) reduces to $\delta_\alpha{}^\beta\delta_A{}^B$ and Eq. (6) to $\delta_{\mu\nu}(1-\delta_{\mu4}\delta_{\nu4})=\delta_{ik}$. We are now in the compact subspace where closure is the same as completeness, for any SU(6) representation. Thus (III) simply follows from the compactness of SU(6), and the proof of (IV) in relation to (III) is also clear.

From this it follows that the parameters characterizing the departures from unitarity are all typically of the type v/c, where v is the velocity of spin-carrying particles involved in the process.[6]

In the foregoing we have exclusively dealt with two-particle unitarity. In a process like meson-baryon scattering there arises also the interesting question of one-particle unitarity when one extrapolates to the baryon pole. We are indebted to N. Khuri for raising this point. The summation over intermediate states now includes all states of the $\underline{56}$ with the appropriate charge, hypercharge, and spin projections. It can be shown that also in this case unitarity is not strictly observed. However, by going to the nonrelativistic limit unitarity is again recovered, and by taking the residues at the baryon pole one obtains the following coupling-constant sum rules from the nonrelativistic baryon-meson vertex[3]:

$$\begin{aligned}&g^2(\pi^+pN^{*++})-g^2(\pi^-pn)-g^2(\pi^-pN^{*0})\\&=\tfrac{1}{2}[-g^2(K^-p\Lambda)-g^2(K^-p\Sigma^0)-g^2(K^-pY^{*0})]\\&=-g^2(K^-n\Sigma^-)-g^2(K^-nY^{*-}).\end{aligned} \tag{7}$$

These relations are of course intimately connected with the Johnson-Treiman relations[7]

$$\begin{aligned}f(\pi^+p)-f(\pi^-p)&=\tfrac{1}{2}[f(K^+p)-f(K^-p)]\\&=f(K^+n)-f(K^-n),\end{aligned} \tag{8}$$

where f denotes the elastic-scattering amplitude in the forward direction.

At this point we digress and ask for the conditions under which Eq. (8) is valid. We have noted earlier[1] that Eq. (8) follows from $SU(12)_{\mathcal{L}}$. Closer inspection shows that Eq. (8) is valid under much more general conditions, however. Sufficient conditions to obtain Eq. (8) are the following. Consider the two distinct boosts[1] $\underline{56}[(\frac{1}{2})^3,0]$ and $\underline{56}[(\frac{1}{2})^2,\frac{1}{2}]$ for the $\underline{56}$, and $\underline{35}[\frac{1}{2},\frac{1}{2}]^-$ and $\underline{35}[(\frac{1}{2})^2,0]^-$ for the $\underline{35}$. Make up all covariant four-point function "$B^\dagger B\mathfrak{M}\mathfrak{M}$" following the same general rules of contraction given earlier for the three-point case.[8] Then any linear combination of this set of four-point functions yields Eq. (8). The proof is quite simple if one remembers that covariance allows us without loss of generality to compute in the rest frame of the proton.

A second side remark: We may ask likewise whether structures for the vertex more general than given by $SU(12)_{\mathcal{L}}$ can still reproduce all the successes of SU(6). We have found the following. Consider the vertex[9]

$$B_{\lambda\mu\rho}^{\dagger}(p_1)({}^{(1)}\mathfrak{M}+\xi\,{}^{(2)}\mathfrak{M})_{\nu}^{\ \rho}(q)B^{\lambda\mu\nu}(p_2), \qquad (9)$$

which reduces to the ($\underline{143}$, $\underline{364}$) coupling for $\xi=1$. One shows that this vertex yields $D/F = \frac{3}{2}$ for pseudoscalar mesons, independent of ξ. If one assumes that the photon-baryon vertex is dominated by the appropriate one-particle states belonging to the $\underline{35}$, one finds for the magnetic moment of the proton

$$\mu(p)=[1+\xi\cdot 2M_{00}/\mu_{00}], \qquad (10)$$

in units $e\hbar/2M_{00}c$. For $\xi=1$, one has the expression first given by Delbourgo, Salam, and Strathdee.[2] The determination of ξ deserves further study.[10]

We return now to Eq. (8). The transition from Eq. (8) to total cross sections via the optical theorem requires the implementation of unitarity which (even in the forward direction) cannot be done for completed SU(6), except in the static limit. It is in this limit that we can relate the four-point relation (8) to the three-point relation (7).

To the extent that the Johnson-Treiman relation is successful one may be further encouraged to believe that completed SU(6) gives a leading, albeit nonunitary, approximation to an as yet unknown unitary theory.

We conclude with the following comments.

(a) The clash between completed SU(6) and local Lagrangian field theory has been emphasized before.[1,3,4] The above counter examples pinpoint the issue on a more physical level. In particular, we bypass in this way the delicate problems associated with the magnitude of renormalization constants.

(b) The problems associated with noncompact groups such as $SU(12)_{\mathcal{L}}$ are, of course, not at all surprising. Conservation of probability would require the realization of unitary representations in Hilbert space. Since the operations of the group do not act on the momenta separately, such realizations are not possible without the introduction of infinite supermultiplets. This last point has in particular been emphasized by Coleman.[11]

(c) More generally, the view expressed here and in foregoing papers[1,3] that the completion of SU(6) can only represent a leading approximation is in harmony with many general theorems on the synthesis of internal symmetries and the Lorentz group. Nor does this view, based entirely on the conventional rules of quantum mechanics in four-dimensional Minkowski space, preclude a rigorous covariant synthesis of spin and unitary spin in a world of 36 dimensions.[12]

(d) Staying within the conventional theory, the rationale for the recipes embodied in $SU(12)_{\mathcal{L}}$ or other forms of SU(6) completion has been in first instance to provide a framework for the comparison with experiment. A prime task, therefore, now appears to be the dynamical recognition of those parameters whose "smallness" may justify, wherever necessary, the effective S matrix as a leading approximation.

We wish to thank Professor C. N. Yang for a stimulating discussion.

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[5]We are indebted for the private communication of this result which was obtained by studying the elastic scattering "($\underline{1}$) + ($\underline{143}$) → ($\underline{1}$) + ($\underline{143}$)," where the numbers denote $SU(12)_{\mathcal{L}}$ meson representations.

[6]Such a parameter is, of course, meaningful only in situations where the momenta are restricted by a bounded phase space, as in the calculation of absorptive parts of collision amplitudes. No parameter is available for the discussion of the full amplitudes unless one is willing to conjecture the damping mentioned in reference 3.

[7]K. Johnson and S. B. Treiman, Phys. Rev. Letters 14, 189 (1965).

[8]We refer in particular to the discussion of Eqs. (10) and (11) in reference 1.

[9]All symbols are as defined in reference 1; see especially Eq. (12).

[10]To indicate orders of magnitude, using the central masses given by M. A. B. Bég and V. Singh [Phys. Rev. Letters 13, 418, 681(E) (1964)] and $\xi=\frac{1}{2}$, one has $\mu(p) = 2.73$. ξ corresponds to f_T/f_V in II, reference 3. As was already noted in reference 1, the magnetic moment ratios persist under conditions even more general than Eq. (9).

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SELECTION RULES IN RELATIVISTIC SU(6) THEORIES

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In a previous paper selection rules were obtained from SU(6) symmetry and found to give a consistent description of a number of known experimental facts.[1] Recently, some of these selection rules have been found to hold in particular relativistic SU(6) formulations.[2,3] We wish to show that the original derivation is valid relativistically under certain general assumptions shared by most of the relativistic formulations, and that all of the results obtained[1] are valid relativistically. In particular, the following decays and reactions are all forbidden:

$$\varphi \to \rho + \pi, \tag{1a}$$

$$\varphi \to 3\pi \text{ or } 5\pi, \tag{1b}$$

$$\pi + N \to \varphi + N + n\pi, \tag{1c}$$

$$N + N \to \varphi + N + N + n\pi, \tag{1d}$$

$$\bar{N} + N \to \varphi + n\pi, \tag{1e}$$

where n is any number.[4] Also the production of strange particles in πN, NN, and $\bar{N}N$ reactions is inhibited by a factor of at least four, on the average. The experimental validity of these selection rules can therefore be considered as evidence in favor of SU(6) symmetry, but cannot be used to distinguish between different relativistic formulations which should all give the same result. In particular, the results (1) should be obtained both from theories which embed SU(6) in a higher group including both SU(6) and the Lorentz group,[2] and theories which break SU(6) symmetry using kinetic-energy spurions.[3]

The selection rules were obtained using SU(2) subgroups of SU(6) called quark spins, $S_{p'}$, $S_{n'}$, and $S_{\lambda'}$, defined, respectively, as the total spin of the p', n', and λ' quarks in any multiquark state. These rules are still valid in any relativistic theory which satisfies the following condition:

The λ'-quark spin $S_{\lambda'}$ is conserved in all reactions involving only particles at rest and particles of finite momentum which contain no λ' quarks in a quark model. It is also conserved in reactions also involving particles of strangeness ±1 which contain one λ' quark or antiquark, where such particles have $S_{\lambda'} = \frac{1}{2}$, but the <u>orientation</u> of the quark spin may be different for a particle of finite momentum than for a nonrelativistic particle. Analogous conservation laws apply to $S_{p'}$ and $S_{n'}$.

It is easily seen that this condition is satisfied by theories of both types mentioned above.[2,3] The essential point is that the number of λ' quarks and antiquarks in a state cannot be changed by a Lorentz transformation nor by the application of the symmetry-breaking kinetic-energy operator. The only effect of such transformations on a multiquark state is to rotate and recouple the spins of the existing quarks. The transition matrix elements for all the reactions (1) vanish in nonrelativistic SU(6) because the φ has $S_{\lambda'} = 1$ and all the other particles contain no λ' quarks and have $S_{\lambda'} = 0$. This remains unchanged by Lorentz transformations of the individual particle states to finite momenta and by adding an arbitrary number of kinetic-energy "spurions" to the vertex function. These transformations do not affect the $S_{\lambda'} = 1$ of the φ, if the reactions are analyzed in a Lorentz frame in which the φ is at rest. They also do not affect the $S_{\lambda'} = 0$ of the other particles, since neither Lorentz transformations nor operation with kinetic-energy operators can produce states having $S_{\lambda'} \neq 0$, as this requires the creation of λ' quarks. Since the theory is Lorentz invariant, forbidding the reactions (1) in a particular Lorentz frame is sufficient to forbid it in all Lorentz frames.[5]

The selection rule against strange-particle production is based on similar considerations and also on the property that a state of two strange particles contains a λ' quark and antiquark, with uncorrelated spins of $\frac{1}{2}$, so that the probability of total $S_{\lambda'} = 0$ is only $\frac{1}{4}$. This is also unaffected by Lorentz transformations