QUARK MODEL IN MOMENTUM SPACE AND HIGHER SYMMETRIES*

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The spin–unitary-spin supermultiplet theory of Gürsey and Radicati[1] and of Sakita[2] has aroused much interest on account of its many impressive results.[3] The fact, as originally recognized by Sakita, that the theory has been formulated really only in a nonrelativistic sense raises questions of interpretation as to the manner in which this theory is to be understood as some kind of limit of a relativistic theory. The attention of many authors is currently focused on these questions. One approach has been suggested by Bég and Pais,[4] wherein the underlying SU(6) symmetry group is considered to hold only in the zero-momentum limit as a "boundary condition" on a relativistic theory whose invariance group is just the direct product of the Lorentz group and the internal symmetry group SU(3). The way of incorporating these constraints is given by Bég and Pais[4] in the form of a relativistic "completion" procedure. An entirely different approach is that of Marshak and Okubo,[5,6] who start with a three-field quark model and look for possible underlying higher unitary groups when one mixes the so-called Casimir projections of the three fields using the γ_4 operator of the Dirac theory. The algebra of the groups underlying such three-field models has also been discussed by Bardakci et al.,[7] by Salam,[8] and by Delbourgo, Salam, and Strathdee,[9] and, in still another form, by Feynman, Gell-Mann, and Zweig.[10]

It appears to us that in such discussions a momentum-space approach can be quite illuminating. The reason is that, after all, it is only in the zero-momentum limit that one has the encouraging results of the SU(6) supermultiplet theory. We further feel that a natural framework for their discussions is provided by the Foldy-Worthuysen[11] (FW) representations for spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ particles. This is due to the fact that, in the FW representations, on the one hand, the "completion" procedure of Bég and Pais is easily constructed, and on the other the γ_4 projections used by Marshak and Okubo[5,6] have a direct meaning as giving the positive- and negative-energy components even when the momentum is nonzero.

In this note, we take the approach outlined above. In Sec. I we give a short resume of the FW representation and indicate briefly its use in the relativistic completion procedure of Bég and Pais.[4] In Sec. II, we then go on to consider the symmetries underlying the three-quark model in a momentum space formulation. As already noted by Okubo and Marshak,[6] we find that the free-field Hamiltonian is invariant under a U(12) group. It is also shown how one may write a four-fermion interaction which is U(6)-invariant in the limit of large quark mass, but otherwise contains parts which reduce this invariance to just U(3).

I. Resume of FW representation. –(a) Spin-$\frac{1}{2}$ Dirac theory. The Dirac equation is in standard notation

$$i\frac{\partial\psi}{\partial t}=(\vec{\alpha}\cdot\vec{p}+\beta m)\psi. \tag{1}$$

In the FW representation obtained by applying the unitary transformation[11]

$$\psi\to\varphi=e^{iS}\psi, \tag{2}$$

where

$$e^{iS}=\frac{(E_p+m)+\vec{\gamma}\cdot\vec{p}}{[2E_p(E_p+m)]^{1/2}};\quad E_p=+(\vec{p}^2+m^2)^{1/2},\quad \vec{\gamma}=\beta\vec{\alpha},$$

the Dirac equation becomes

$$i\partial\varphi/\partial t=\beta E_p\varphi. \tag{3}$$

In the usual Pauli representation of Dirac matrices, the normalized FW spinors are simply

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad\begin{pmatrix}0\\1\\0\\0\end{pmatrix},$$

for positive energy, and

$$\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad\begin{pmatrix}0\\0\\0\\1\end{pmatrix},$$

for negative energy. These spinors do not depend on the momentum and are trivially obtained from constant two-component Pauli spinors. This is the reason why the Bég and Pais "relativistic completion"[4] for spin-$\frac{1}{2}$ particles will be trivially obtainable in the FW representation. To go to the usual Dirac representation one simply applies the transformation e^{-iS}.

Now, in the Dirac theory the spin and orbital angular momentum operators are not separately constants of motion. However, one can define the so-called mean-spin operators[11,12] $\vec{\Sigma}$ and mean orbital angular momentum $\vec{L}$, which are separately constants of motion. These are given by

$$\vec{\Sigma} = \vec{\sigma} - i\beta(\vec{\alpha}\times\vec{p})/E_p - \vec{p}\times(\vec{\sigma}\times\vec{p})/[E_p(E_p+m)], \quad (4)$$

$$\vec{L} = \vec{X}\times\vec{p}, \quad (5)$$

where

$$\vec{X} = \vec{x} + i\beta\vec{\alpha}/(2E_p) - i[\beta(\vec{\alpha}\cdot\vec{p})\vec{p} - i(\vec{\Sigma}\times\vec{p})|\vec{p}|]/[2E_p(E_p+m)|\vec{p}|].$$

The advantage of the FW representation is again seen in that these operators have the simple form of the usual spin operator $\vec{\sigma}$ and the orbital angular momentum operator

$$\vec{x}\times\vec{p} = i(\partial/\partial\vec{p})\times\vec{p}.$$

(b) Spin-$\frac{3}{2}$ theory. The Rarita-Schwinger[13] equation for spin $\frac{3}{2}$ has been written in a reduced canonical form by Moldauer and Case[16] as

$$i\frac{\partial}{\partial t}\Psi_{(3/2)} = H_{(3/2)}\Psi_{(3/2)}, \quad (6)$$

where $\Psi_{(3/2)}$ is an eight-component spinor (the four components required to describe a spin-$\frac{3}{2}$ particle are naturally doubled since both positive and negative energy states appear in the relativistic theory). Using a theorem of Case,[15] an FW-type transformation can be carried through, namely:

$$\Psi_{(3/2)} \rightarrow U\Phi_{(3/2)}, \quad (7)$$

whereby $\Phi_{(3/2)}$ satisfies the equation

$$i\frac{\partial}{\partial t}\Phi_{(3/2)} = \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix} E_p \Phi_{(3/2)};$$

$$i\frac{\partial}{\partial t}\Phi_{(3/2)} = \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix} E_p \Phi_{(3/2)}; \quad (8)$$

$$E_p = (\vec{p}^2 + m^2)^{1/2}.$$

The transformation U is given by

$$U = 1_2 \otimes \cos\tfrac{1}{2}\varphi - i\rho_2 \otimes \sin\tfrac{1}{2}\varphi, \quad (9)$$

where 1_2 is 2×2 unit matrix and ρ_2 a 2×2 Pauli matrix,

$$\rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

acting between the positive- and negative-energy components of the eight-component spinor, and where

$$\tan\varphi = \frac{p}{m}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -\dfrac{(\frac{4}{3})p^2+m^2}{(8/3)p^2+3m^2} & 0 & 0 \\ 0 & 0 & \dfrac{(\frac{4}{3})p^2+m^2}{(8/3)p^2+3m^2} & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}; \quad p = |\vec{p}|. \quad (10)$$

The completion procedure of Bég and Pais[4] for the spin-$\frac{3}{2}$ baryon states occurring in the 56-dimensional representation of SU(6) is easily carried out in the FW representation. Thus, for example, corresponding to the nonrelativistic spin-$\frac{3}{2}$ spinor,

$$\chi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

the relativistically completed eight-component spinor in the FW representation is simply

$$\begin{pmatrix} \chi \\ 0_4 \end{pmatrix}, \quad 0_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and is independent of the momentum. In the usual representation, one obtains the momentum-dependent spinor by applying the inverse transformation U. In this manner, one can easily carry through the completion procedure.

II. Three-field model. – Let ψ_{ip}, $i=1,2,3$, be a U(3) triplet of Dirac spinor operators in momentum space. Making use of the constant mean-spin operators $\vec{\Sigma}$ introduced in Eq. (4) and the nine generators of U(3) for the triplet representation,[16] namely, λ_A $(A=0,1,\cdots,8)$, we may construct the 36 quantities

$$\int d^3p\,\psi_p^\dagger\lambda_A\otimes\Sigma_\mu\psi_p \qquad (11)$$

$$(A=0,1,\cdots,8,\ \mu=0,1,2,3),$$

where

$$\psi_p=\begin{pmatrix}\psi_{1p}\\ \psi_{2p}\\ \psi_{3p}\end{pmatrix},$$

and

$$\Sigma_0=1_4.$$

These may be used to generate a U(6) group, which we call the group $U_\Sigma(6)$ in order to distinguish it from the various U(6) groups being discussed in the literature. The relation of our $U_\Sigma(6)$ group to the Sakita-Gürsey-Radicati group is made transparent by going to the FW representation in which the above 36 generators simply become

$$A_r=\int d^3p\,\varphi_p^\dagger\lambda_A\otimes\sigma_\mu\varphi_p \qquad (12)$$

$$(A=0,1,\cdots,8,\ \mu=0,1,2,3,\ r=0,1,\cdots,35),$$

where

$$\sigma_0=1.$$

Let us look at the free-field Hamiltonian

$$H_0=\sum_{i=1}^{3}\int d^3p\,\psi_{ip}^\dagger(\vec{\alpha}\cdot\vec{p}+\beta m)\psi_{ip}, \qquad (13)$$

which in the FW representation takes the simple form

$$H_0=\sum_i\int d^3p\,E_p\varphi_{ip}^\dagger\gamma_4\varphi_{ip}, \qquad (14)$$

and is clearly seen to commute with the 36 generators A_r, for γ_4 commutes with $\vec{\sigma}$. Thus H_0 is invariant under $U_\Sigma(6)$. Also invariant under this group is the mean orbital angular momentum operator

$$\vec{L}=\sum_i\int d^3p\,\psi_{ip}^\dagger(\vec{X}\times\vec{p})\psi_{ip}$$
$$=\sum_i\int d^3p\,\varphi_{ip}^\dagger[i(\partial/\partial\vec{p})\times\vec{p}]\varphi_{ip}. \qquad (15)$$

We may also consider the positive- and negative-energy projections of the φ_{ip}, given, respectively, by

$$\xi_{ip}=\tfrac{1}{2}(1+\gamma_4)\varphi_{ip}, \qquad (16)$$

and

$$\eta_{ip}=\tfrac{1}{2}(1-\gamma_4)\varphi_{ip}.$$

In terms of the ξ_{ip} and η_{ip}, the free-field Hamiltonian can be written as

$$H_0=\sum_i\int d^3p(\xi_{ip}^\dagger\xi_{ip}-\eta_{ip}^\dagger\eta_{ip}), \qquad (17)$$

and is thus seen to be invariant under a group $U_\sigma^{(1)}(6)\otimes U_\sigma^{(2)}(6)$, where the groups $U_\sigma^{(1)}(6)$ and $U_\sigma^{(2)}(6)$ are generated, respectively, by

$$A_r^{(1)}=\int\xi_p^\dagger\lambda_A\otimes\sigma_\mu\xi_p d^3p,\quad \xi_p=\begin{pmatrix}\xi_{1p}\\ \xi_{2p}\\ \xi_{3p}\end{pmatrix};$$

$$A_r^{(2)}=\int\eta_p^\dagger\lambda_A\otimes\sigma_\mu\eta_p d^3p,\quad \eta_p=\begin{pmatrix}\eta_{1p}\\ \eta_{2p}\\ \eta_{3p}\end{pmatrix}. \qquad (18)$$

The generators A_r of $U_\Sigma(6)$ can be expressed as

$$A_r=A_r^{(1)}+A_r^{(2)}. \qquad (19)$$

We may go even a step further, in that we anticommute $\eta_{ip}^\dagger$ and η_{ip} in H_0 and drop the infinite constant due to the anticommutator, obtaining

$$H_0'=\int d^3p(\xi_p^\dagger\xi_p+\eta_p^T\eta_p^*). \qquad (20)$$

Here η_p^T is the transpose of η_p, and η_p^* is the complex conjugate η_p. Thus H_0' is in fact invariant under a group U(12),[6] whose transformations act on the 12-component spinor operator

$$\Phi_p=\begin{pmatrix}\xi_p\\ \eta_p^*\end{pmatrix}.$$

This way of searching for higher group symmetries like U(6)⊗U(6) and U(12) in a three-field model has also been discussed in a different approach by Okubo and Marshak.[5,6] We may emphasize here what was already mentioned in the beginning of this note, that the γ_4 projection in momentum space employing the FW representation has a very clear meaning.

The question to be considered is whether we can write an interaction Hamiltonian in momentum space, which has any of these higher symmetries. Let us write a four-fermion pseudoscalar (P) interaction in momentum space which would be equivalent to a local x-space interaction, namely,

$$H_1 = \sum_{i,j=1}^{3} \int d^3p_1 d^3p_2 d^3p_3 (\psi_{ip_1}{}^\dagger \gamma_4\gamma_5 \psi_{ip_2})(\psi_{jp_3}{}^\dagger \gamma_4\gamma_5\psi_{jp_4}), \tag{21}$$

where we take

$$\vec{p}_1 + \vec{p}_3 = \vec{p}_2 + \vec{p}_4.$$

Since

$$e^{iS(p_1)}\gamma_4\gamma_5 e^{-iS(p_2)} = a\gamma_4\gamma_5 + b\gamma_4\gamma_5 i\vec{\sigma}\cdot(\vec{p}_1\times\vec{p}_2) + ci\vec{\sigma}\cdot\vec{p}_1 + di\vec{\sigma}\cdot\vec{p}_2, \tag{22}$$

where a, b, c, and d are functions of p_1, p_2, and m, we can write H_1 in the FW representation in the form

$$H_1 = \sum_{i,j=1}^{3} \int d^3p_1 d^3p_2 d^3p_3 [\varphi_{ip_1}(a\gamma_4\gamma_5 + b\gamma_4\gamma_5 i\vec{\sigma}\cdot(\vec{p}_1\times\vec{p}_2) + ci\vec{\sigma}\cdot\vec{p}_1 + di\vec{\sigma}\cdot\vec{p}_2)\varphi_{ip_2}]$$
$$\times[\varphi_{jp_3}(a'\gamma_4\gamma_5 + b'\gamma_4\gamma_5 i\vec{\sigma}\cdot(\vec{p}_3\times\vec{p}_4) + c'i\vec{\sigma}\cdot\vec{p}_3 + d'i\vec{\sigma}\cdot\vec{p}_4)\varphi_{jp_4}]. \tag{23}$$

Here a', b', c', and d' are the same functions of p_3, p_4, and m as a, b, c, and d are of p_1, p_2, and m. This interaction contains a part

$$H_1' = \sum_{i,j=1}^{3} \int d^3p_1 d^3p_2 d^3p_3 (\varphi_{ip_1} a\gamma_4\gamma_5 \varphi_{ip_2})(\varphi_{jp_3} a'\gamma_4\gamma_5\varphi_{jp_4}), \tag{24}$$

which is invariant under $U_\Sigma(6)$ (since γ_4 and γ_5 commute with $\vec{\sigma}$). The remaining part of H_1 breaks $U_\Sigma(6)$ symmetry, and is invariant only under the group U(3).

If the mass m of the basic triplet ψ_i is taken to be very large, then one sees easily that the orders of the coefficients a, a', b, b', etc., are

$$a, a' \sim 1,$$
$$c, c', d, d' \sim 1/m,$$
$$b, b' \sim 1/m^2.$$

Thus we see that in the limit of very large quark mass, the dominant term of H_1 is $U_\Sigma(6)$ invariant. The next order term has the form

$$H_1'' \simeq \sum_{i,j=1}^{3} \int d^3p_1 d^3p_2 d^3p_3 \times[(\varphi_{ip_1}\gamma_4\gamma_5\varphi_{ip_2})(\varphi_{jp_3}\vec{\sigma}\cdot\vec{p}'\varphi_{jp_4}) + (\varphi_{ip_1}\vec{\sigma}\cdot\vec{p}\varphi_{ip_2})(\varphi_{ip_3}\gamma_4\gamma_5\varphi_{jp_4})], \tag{25}$$

which behaves as a component of the regular tensor operator of $U_\Sigma(6)$ (i.e., belongs to its 36-dimensional representation), and thus breaks the $U_\Sigma(6)$ symmetry in a well-defined way while preserving the lower U(3) symmetry. We see that, in contrast to the results of other authors,[5-7,9] it is not here the kinetic-energy part of the free-field Hamiltonian, but a part of the interaction itself that breaks the $U_\Sigma(6)$ symmetry.

It may be remarked here that, since H_1 is the momentum-space equivalent of a local relativistically invariant interaction, the corresponding S-matrix element will be crossing symmetric.

Before concluding this note, we should like to mention that if we do not wish to stick to the requirements of "locality" of the interaction, then we can write a $U_\Sigma(6)$-symmetric interaction in the momentum space, namely,

$$H_2 = \sum_{i,j=1}^{3} \int d^3p (\psi_{ip}{}^\dagger\gamma_4\gamma_5\psi_{ip})(\psi_{jp}{}^\dagger\gamma_4\gamma_5\psi_{jp}),$$
$$= \sum_{i,j=1}^{3} \int d^3p (\varphi_{ip}{}^\dagger\gamma_4\gamma_5\varphi_{ip})(\varphi_{jp}{}^\dagger\gamma_4\gamma_5\varphi_{jp}). \tag{26}$$

However, it must be recognized that for such a theory the Lorentz invariance is clear only when it is looked upon as an unquantized C-number theory.

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Note added in proof.—After this note had been written we received a preprint of a paper by K. T. Mahanthappa and E. C. G. Sudarshan, in which these authors have also employed the operator $\vec{\Sigma}$ for constructing a U(6) algebra.

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DECAY $Y_1^*(1660) \to Y_0^*(1405) + \pi$†

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Thus far, particles of the same known spin and parity have been successfully assigned into SU(3) multiplets by using the Gell-Mann–Okubo mass formula.[1] Attempts to classify particles whose spins and parities are not well established into particular multiplets on the basis of the mass formula may lead to contradictory assignments, as in the case of the $Y_1^*(1660)$.[2,3] Additional information on the multiplet assignment of a particle may be derived from its decay modes. Where the SU(3)-breaking interactions can be neglected, SU(3) gives definite predictions of branching ratios and selection rules for the decay of a member of a given multiplet into members of other multiplets.[4] We report here experimental evidence for the decay $Y_1^*(1660) \to Y_0^*(1405) + \pi$, which can be used as evidence that the $Y_1^*(1660)$ is a member of an octet if $Y_0^*(1405)$ is assumed to be a unitary singlet.

The data on the $Y_1^*(1660)$ or[5] $\Sigma(1660)$ decay modes were obtained from an analysis of the following reactions:

$$K^- + p \to \Sigma^+ + \pi^+ + \pi^- + \pi^-, \quad (1)$$

$$K^- + p \to \Sigma^- + \pi^+ + \pi^+ + \pi^-, \quad (2)$$

$$K^- + p \to \Lambda + \pi^+ + \pi^0 + \pi^-. \quad (3)$$

Reactions (1) and (2) give information on the decay $\Sigma(1660) \to \Lambda(1405) + \pi$, and Reaction (3) is used to give an upper limit on the amount of $\Sigma(1660) \to \Sigma(1385) + \pi$.