ROUGH COMPUTATION OF THE SEPARATION ENERGY OF STRONGLY BOUND NUCLEONS

D. M. Brink
Clarendon Laboratory, Oxford, England

and

N. Sherman*
Department of Physics, University of Michigan, Ann Arbor, Michigan
(Received 12 January 1965)

The results of recent $(e,e'p)$ experiments[1] have re-emphasized the desirability of a reliable nuclear model which is capable of connecting the shell-model description of nuclear levels with binding energies.[2] In particular, the spectrum of protons ejected from Al^{27} indicates that it is composed of three groups of protons, in agreement with a shell-model picture of the struck nucleus. If one associates the energy of each group with a corresponding level in the shell-model description of Al^{27}, then the energy required to remove protons from the $1s$ shell is about 60 MeV. If, as expected,[3] the separation is accomplished adiabatically (i.e., without disturbing any of the nucleons other than the struck proton), then the depth of the shell-model potential must be greater than 60 MeV. Although calculations of the properties of infinite nuclear matter indicate effective potentials greater than 60 MeV, for strongly bound nucleons,[4] the well depth for finite nuclei is usually considered to be more like 45-50 MeV.[5] The purpose of this report is to describe a rough but reasonable estimate of the adiabatic separation energy of $1s$ and $1p$ nucleons.

The calculation has several attractive properties: (a) The assumptions and approximations are explicit so that refinements can be made if desired; (b) it requires only a few parameters for the description of many data; (c) it may serve as an efficient first step in a more ambitious self-consistent calculation.

Let us consider first the problem of estimating the adiabatic separation energy of a nucleon in the $1s$ level in any nucleus between, say, Li^7 and Ca^{40}. (The methods used can also be applied to heavier nuclei.) The framework of the model is characterized by three general assumptions:

(i) The nuclear Hamiltonian is that for a system of A nucleons subject to two-body forces,

$$H=\sum_i t_i+\tfrac{1}{2}\sum_{i\neq j} v_{ij}. \tag{2.1}$$

(ii) The two-body internucleon potential v_{ij} is some appropriate empirical expression chosen to fit the nucleon-nucleon scattering data; it should be charge independent and either sufficiently nonsingular to permit computation of two-particle matrix elements using single-particle wave functions, or subject to treatment that permits computation of matrix elements (e.g. the Scott-Moszkowski[6] separation method).

(iii) The nuclear wave function can be described by a single Slater determinant (although this restriction can be relaxed).

Besides these general assumptions we make a number of specific approximations:

(iv) v_{ij} is a central, charge-independent interaction.

(v) Harmonic-oscillator single-particle wave functions are used to construct the Slater-determinant many-particle wave function.

(vi) The oscillator energy $\hbar\omega$, hence the harmonic-oscillator radius parameter $b=(\hbar/m\omega)^{1/2}$, is determined from the mean square radius of the nuclear charge distribution.

(vii) Each nucleon in a partially filled shell contributes the same amount to the separation energy of a $1s$ or $1p$ nucleon as it would in a completely filled shell. This is equivalent to assuming that the interaction of a tightly bound nucleon with nucleons in the outermost unfilled shell is not sensitive to the coupling of the nucleons in the unfilled shell.

If we make these assumptions then the separation energy of a $1s$ or a $1p$ nucleon in a nucleus lighter than Ca^{40} may be expressed in terms of four parameters $V_n (n=0,1,2,3)$ which are the interaction energies between pairs of nucleons in states of relative motion with principal quantum number n [corresponding to an oscillator energy $(n+\frac{3}{2})\hbar\omega$ of relative motion] averaged over angular momenta. We have

$$V_0=V_{00},\quad V_1=V_{01},\quad V_2=\tfrac{1}{6}(V_{10}+5V_{02}),$$
$$V_3=\tfrac{1}{10}(3V_{11}+7V_{03});$$

where

$$V_{nl}=\langle nl|c_s V_s+c_t V_t|nl\rangle. \tag{1}$$

V_S and V_t are the singlet and triplet parts of the nucleon-nucleon interaction; $c_S = c_t = \frac{1}{2}$ if l is even, and $c_S = \frac{1}{10}$, $c_t = 9/10$ if l is odd; and $|nl\rangle$ is a harmonic-oscillator function of relative motion with principal quantum number n and orbital angular momentum l. Using assumptions (i) and (iii), the adiabatic separation energy ($-E_\alpha$) of a nucleon in the state $|\alpha\rangle$ is given by

$$E_\alpha = \langle\alpha|t|\alpha\rangle + \sum_{\beta=1}^{A} (\langle\alpha\beta|v|\alpha\beta\rangle - \langle\alpha\beta|v|\beta\alpha\rangle). \quad (2)$$

In this equation β stands for spin and isobaric spin as well as for orbital quantum numbers. If we restrict the discussion for the moment to the doubly closed-shell nuclei O^{16} and Ca^{40}, the sums over spin and isobaric spin can be carried out, and we are left with a sum over orbital quantum numbers.

The sum can then be simplified by using approximation (iv), which allows use of the Talmi-Moshinsky[7] techniques of expressing two-nucleon wave functions in terms of center-of-mass and relative-motion wave functions. For example, one can use an equivalent potential[8] expressed in terms of projection operators q_{nl} on states of relative motion of the two nucleons,

$$v \to v_{\text{eq}} = \sum_{nl} V_{nl} q_{nl},$$

where V_{nl} are defined in Eq. (1) and

$$\langle n'l'|q_{nl}|n''l''\rangle = \delta(n',n)\delta(n'',n)\delta(l'',l)\delta(l'',l).$$

After making these steps, E_α in Eq. (2) becomes

$$E_\alpha = \langle\alpha|t|\alpha\rangle + \sum_{b}^{\text{occ}} \sum_{nl} V_{nl}(4-A_{nl})\langle\alpha b|q_{nl}|\alpha b\rangle, \quad (3)$$

where the coefficients $A_{nl} = (-1)^l$ comes from the exchange term in Eq. (2). The matrix elements of q_{nl} are pure numbers which can be expressed in terms of Moshinsky transformation coefficients or evaluated directly using the techniques described in reference 8.

Finally, for nuclei with unfilled shells we use approximation (vii), which implies that n_p (n_{sd}) nucleons in an unfilled $1p$ shell ($2s$-$1d$ shell) contribute a fraction $n_p/12$ ($n_{sd}/24$) of the contribution of the filled shell to the binding energy of a strongly bound nucleon.

It should be noted that this calculation makes use of the special properties of harmonic-oscillator wave functions, and many of the steps could not be carried through with other wave functions.

The model described above leads to the following simple expression for the adiabatic separation energy ($-E_{1s}$) of nucleons in the $1s$ shell of a nucleus between Li^7 and Ca^{40}:

$$E_{1s} = t_{1s} + 3V_0 + n_p(\tfrac{3}{8}V_0 + \tfrac{5}{8}V_1) + n_{sd}(\tfrac{3}{16}V_0 + \tfrac{5}{8}V_1 + \tfrac{3}{16}V_2), \quad (4)$$

where $t_{1s} = \frac{3}{4}\hbar\omega$ = expectation value of the kinetic energy of a nucleon in the $1s$ state of a harmonic oscillator, n_p = the number of nucleons in the $1p$ shell, and n_{sd} = the number of nucleons in the $2s$-$1d$ shell. Similarly, the adiabatic separation energy ($-E_{1p}$) of nucleons in the $1p$ shell is found to be

$$E_{1p} = t_{1p} + (\tfrac{3}{2}V_0 + \tfrac{5}{2}V_1) + n_p[\tfrac{1}{4}V_0 + (5/12)V_1 + \tfrac{1}{4}V_2] + n_{sd}[\tfrac{5}{32}V_0 + (35/96)V_1 + \tfrac{7}{32}V_2 + (25/96)V_3], \quad (5)$$

where $t_{1p} = (5/4)\hbar\omega$ = expectation value of the kinetic energy of a nucleon in the $1p$ state of a harmonic oscillator, and the other quantities have been defined above.

In order to compare Eqs. (4) and (5) with experimental quantities, we must decide on a prescription for calculating the matrix elements which determine V_0, V_1, V_2, and V_3. In principle, these matrix elements are fully determined as soon as the two-body interaction v_{ij} is chosen. In practice, however, the types of v_{ij} which give good agreement with scattering data[9] involve hard cores and therefore require some additional treatment of the singularity (in order to calculate two-body matrix elements), which introduces another source of arbitrariness. For definiteness we have chosen the Kallio-Kolltveit[9] (KK) interaction and have used values for the matrix elements V_{nl} calculated[10] with the Scott-Moszkowski separation method. The same harmonic-oscillator parameter $\hbar\omega = 13.3$ MeV has been used in all calculations. The results are listed in Table I for some nuclei below Ca^{40} for which experimental numbers are also available.[11,12] Figure 1 shows the experimental numbers with theoretical curves calculated[10] from Eqs. (4) and (5) for the KK

Table I. Comparison of theoretical and experimental values of separation energies (in MeV) of $1s$ and $1p$ nucleons[a,b] ($V_0 = -7.6$ MeV, $V_2 = -1.1$ MeV, $V_1 = V_3 = 0$).

Nucleus	$-E_{1s}$ (theory)	$-E_{1s}$ (expt.)	$-E_{1p}$ (theory)	$-E_{1p}$ (expt.)
Li^7	21	25	1	15
B^{10}	30	32	8	11
C^{12}	36	36	12	14
O^{16}	47	34	21	18
Al^{27}	65	59	37	32
S^{32}	73	65	44	32
Ca^{40}	86	•••	55	•••

[a]See reference 11.
[b]See reference 12.

and Hamada-Johnston[13] (HJ) potentials. The Brueckner, Gammel, and Thaler[14] and the KK potentials give almost identical results.

The theoretical results are most sensitive to the value of V_0; a change of 1 MeV in V_0 produces a change of 10 MeV in E_{1S} for Al^{27}. To get reasonable agreement with experiment, V_0 must lie between about 7 and 8 MeV. It is considerably smaller than this for the HJ potential ($V_0 = 6.2$ MeV).

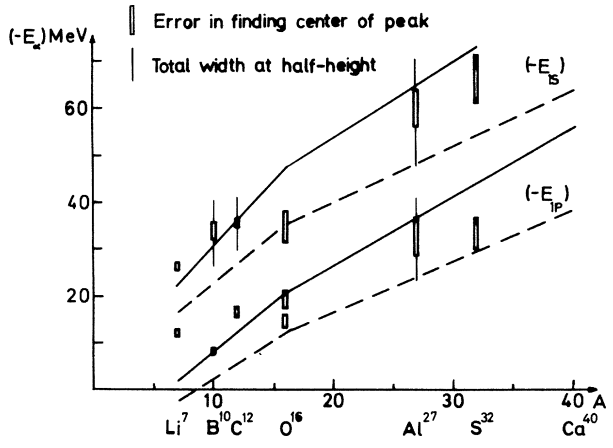

FIG. 1. Theoretical and experimental[11,12] values of the $1s$ and $1p$ separation energies. The solid lines are theoretical values calculated[10] using the Kallio-Kolltveit potential. The dotted lines are calculated[10] from the Hamada-Johnston potential.

It is a pleasure to acknowledge the cooperation of Professor M. Ageno and our colleagues at the Istituto Superiore di Sanità who brought the problem to our attention and provided one of us (N.S.) with warm and most generous hospitality during a sabbatical year in Rome. We would also like to thank Dr. C. W. Wong for making available some of his unpublished results.

*The research reported herein was carried on while this author was on sabbatical leave at the Istituto Superiore di Sanità, Rome, Italy. His work was supported in part by the National Science Foundation.

[1]U. Amaldi, Jr., et al., Phys. Rev. Letters 13, 341 (1964).

[2]J. P. Elliott and A. M. Lane, Encyclopedia of Physics, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, p. 357.

[3]I. Jacob and T. A. Maris, Nucl. Phys. 31, 139, 152 (1962).

[4]S. A. Moszkowski, Encyclopedia of Physics, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, p. 431.

[5]L. A. Sliv and B. A. Volchok, Zh. Eksperim. i Teor. Fiz. 36, 539 (1959) [translation: Soviet Phys.—JETP 9, 374 (1959)].

[6]B. L. Scott and S. A. Moszkowski, Nucl. Phys. 29, 665 (1962).

[7]M. Moshinsky and T. A. Brody, Tables of Transformation Brackets (Monografías del Instituto de Física, Universidad de Mexico, Mexico, D. F., 1960).

[8]D. M. Brink, Nucl. Phys. 40, 593 (1963).

[9]A. Kallio and K. Kolltveit, Nucl. Phys. 53, 87 (1964).

[10]Values for the matrix elements V_{nl} were obtained from C. W. Wong (private communication).

[11]The experimental numbers were compiled by the Sanità group from their own work on C^{12}, Al^{27}, and S^{32} and from $(p,2p)$ experiments reported in the literature. In privately communicating these results to us they point out that for the $(e,e'p)$ results "the errors attributed to the experimental binding energies are roughly estimated maximum errors due to the systematic uncertainties in the energy scale and to the statistical errors in fitting of peaks to the measured spectrum."

[12]The experimental numbers on S^{32} are preliminary results obtained by the Sanità group and presented at the 50th Congresso della Società Italiana di Fisica, Catania, Italy, 26-31 October 1964 (to be published).

[13]T. Hamada and I. D. Johnston, Nucl. Phys. 34, 382 (1962).

[14]K. A. Brueckner and J. L. Gammel, Phys. Rev. 109, 1023 (1958).