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COHERENT STATES AND THE NUMBER-PHASE UNCERTAINTY RELATION*

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Attention has recently been called to the fact that the traditional[1] number-phase "uncertainty relation" connecting the excitation number of an oscillator to the phase angle,

$$\Delta N \Delta \varphi \geq \tfrac{1}{2}, \quad (1)$$

lacks precise meaning for small quantum numbers, since an appropriate Hermitian phase operator $\varphi_{\rm op}$ does not exist.[2,3] Our purpose here is (1) to propose a substitute for Eq. (1) in terms of the (well-defined) operators S and C, and (2) to evaluate the new expression for the so-called coherent states, whose importance has been stressed by Glauber[4] and Sudarshan.[5] It will be shown that for large N the coherent states are minimum-uncertainty number-phase states as well as minimum-uncertainty position-momentum states.[4] Even for very small N the uncertainty product is very small.

We consider a single mode of the radiation field, described by the usual harmonic-oscillator variables a, a^* obeying $[a, a^*] = 1$. The operators S and C are defined[2] in terms of the operators $E_\pm$, whose classical analogs are $\exp(\mp i\psi)$, ψ being the classical phase. Denoting the number operator a^*a by $N_{\rm op}$, we have

$$E_- \equiv (N_{\rm op}+1)^{-1/2}a, \quad E_+ \equiv a^*(N_{\rm op}+1)^{-1/2}. \quad (2)$$

The $E_\pm$ are one-sided unitary, as is shown by the following matrix elements in the number basis:

$$(E_-E_+)_{mn} = \delta_{mn}, \quad (E_+E_-)_{mn} = \delta_{mn} - \delta_{m0}\delta_{n0}. \quad (3)$$

Here the integers m, n, denote the usual number states. The $E_\pm$ are raising and lowering operators:

$$E_\pm |m\rangle = |m \pm 1\rangle. \quad (4)$$

Despite the nonunitary nature of $E_\pm$, we can define Hermitian "sine and cosine" operators[6]

$$S \equiv (1/2i)(E_- - E_+),$$
$$C \equiv \tfrac{1}{2}(E_- + E_+). \quad (5)$$

From the commutation rules

$$[N_{\rm op}, S] = iC,$$
$$[N_{\rm op}, C] = -iS, \quad (6)$$

one can deduce the uncertainty relations $[(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2)]$

$$(\Delta N)^2(\Delta S)^2 \geq \tfrac{1}{4}\langle C\rangle^2,$$
$$(\Delta N)^2(\Delta C)^2 \geq \tfrac{1}{4}\langle S\rangle^2. \quad (7)$$

The proposed relation, which treats the fluctuations in S and C symmetrically, and reduces to (1) in the appropriate limit, is deduced from (7):

$$U \equiv (\Delta N)^2 \frac{[(\Delta S)^2 + (\Delta C)^2]}{[\langle S\rangle^2 + \langle C\rangle^2]} \geq \frac{1}{4}. \quad (8)$$

The coherent states, well known in the study of radiation emitted by a classical current source,[7,8] are defined by[9]

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle;$$

$$|\alpha\rangle = \exp[-(\tfrac{1}{2})\,|\alpha|^2] \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{1/2}}\,|n\rangle. \tag{9}$$

The average number N is $|\alpha|^2$; the states $|n\rangle$ are Poisson distributed and $(\Delta N)^2 = N$. The complex number α has magnitude $N^{1/2}$ and its phase corresponds to the classical phase angle.[10]

The foregoing equations can now be used to give

$$\langle\alpha|S^2|\alpha\rangle = \tfrac{1}{2} - \tfrac{1}{4}e^{-N} - \tfrac{1}{2}e^{-N}N(1-2\xi) \times \sum_{n=0}^{\infty} \frac{N^n}{n![(n+1)(n+2)]^{1/2}}. \tag{10}$$

In Eq. (10) the parameter ξ is defined by

$$\xi = (\mathrm{Im}\alpha)^2/[(\mathrm{Re}\alpha)^2 + (\mathrm{Im}\alpha)^2], \tag{11}$$

and lies in the range $0 \leq \xi \leq 1$. Also,

$$\langle\alpha|S|\alpha\rangle = e^{-N}\,\mathrm{Im}\alpha \sum_{n=0}^{\infty} \frac{N^n}{n!(n+1)^{1/2}}. \tag{12}$$

The corresponding results for C follow by substituting $1-\xi$ for ξ in (10), (11), and (12). [In (12) this means $\mathrm{Im}\alpha \to \mathrm{Re}\alpha$.]

It is interesting to note that

$$\langle\alpha|C^2 + S^2|\alpha\rangle = 1 - \tfrac{1}{2}e^{-N}, \tag{13}$$

which has limits $\frac{1}{2}$ and 1 for N small and large, respectively. The asymptotic expressions (for large N)[11]

$$\sum_{n=0}^{\infty} \frac{N^n}{n![(n+1)(n+2)]^{1/2}} \sim \frac{e^N}{N}\left[1 - \frac{1}{2N} - \frac{3}{8N^2} + \cdots\right], \tag{14}$$

$$\sum_{n=0}^{\infty} \frac{N^n}{n!(n+1)^{1/2}} \sim \frac{e^N}{N^{1/2}}\left[1 - \frac{1}{8N} + \cdots\right], \tag{15}$$

are also useful. In particular, we note the result

$$(\langle\alpha|C|\alpha\rangle)^2 + (\langle\alpha|S|\alpha\rangle)^2 = Ne^{-2N}\left[\sum_{n=0}^{\infty} \frac{N^n}{n!(n+1)^{1/2}}\right]^2 \sim 1 - \frac{1}{4N}. \tag{16}$$

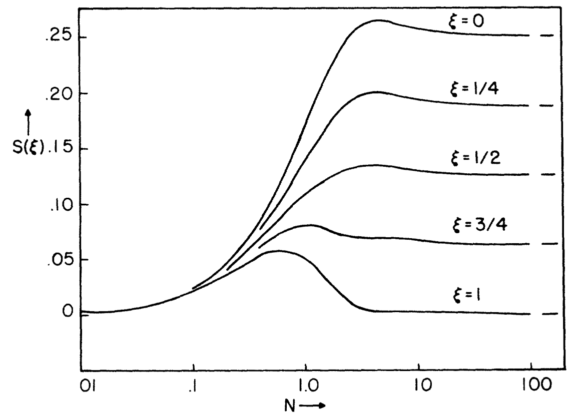

FIG. 1. The uncertainty product $S(\xi) = (\Delta N)^2(\Delta S^2)$ is shown as a function of $(\Delta N)^2 = N$ for various values of the parameter ξ defined in Eq. (11). All expectation values refer to coherent states.

We have evaluated expressions entering in Eqs. (7) and (8) for coherent states having a wide range of mean excitation N. The cumbersome summations were dealt with by means of a CDC 1604 computer, and the results were checked for large and small N by means of the limits discussed above. Figure 1 shows the quantity

$$S(\xi) \equiv (\Delta N)^2(\Delta S)^2 \tag{17}$$

evaluated for coherent states with parameters $\xi = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. Figure 2 shows explicitly that the first uncertainty relation in Eq. (7) is satisfied and closely approached for <u>all</u> N when $\xi = 0$ (real α). Corresponding results

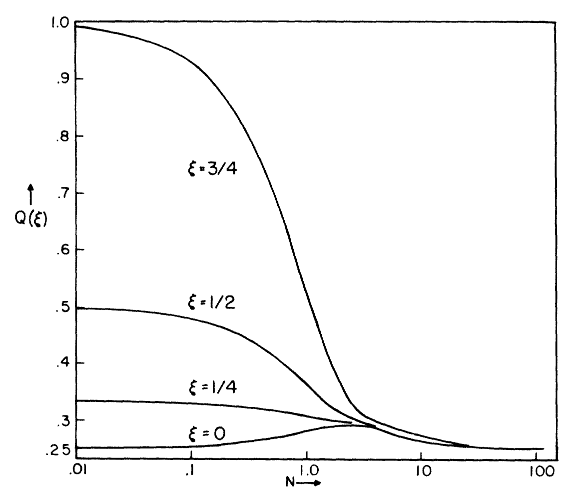

FIG. 2. The quantity $Q(\xi) \equiv S(\xi)/\langle C\rangle^2$ is shown as a function of $(\Delta N)^2 = N$ for various ξ, for the coherent states. According to Eq. (7), $Q(\xi)$ must be larger than $\frac{1}{4}$.

for the second relation in Eq. (7) follow on using the symmetry mentioned after Eq. (12). The maximum is 0.2922 and occurs near $N=2.4$.

Figure 3 shows the uncertainty product U of Eq. (8), which does not discriminate between S and C (U is independent of ξ). The coherent states are seen to be nearly as classical as permitted for all values of mean excitation. In particular, we see that $\frac{1}{2} \geq U \geq \frac{1}{4}$. Brunet[12] has also proposed states having a small uncertainty product. It appears that his definition of phase uncertainty relies on the classical limit and so is only valid for large N. Moreover, the uncertainty product is larger than that which we have found for the coherent states. (Even more important for application is the direct physical meaning of the coherent states.)

Finally we consider the limitation on simultaneous measurements of S and C. From the commutation relation

$$[S, C] = (1/2i)(1-E_+E_-), \tag{18}$$

one finds the relation

$$(\Delta S)(\Delta C) \geq \tfrac{1}{4}e^{-N}. \tag{19}$$

If $N \leq 1$, the equality in Eq. (19) is almost exactly satisfied, for all ξ. This is seen to be in agreement with the fact that as N approaches zero, both ΔS and ΔC go to $\frac{1}{2}$. For large N, the left-hand side decreases as

$$\Delta S \Delta C \sim \left[\frac{1}{16N^2}\xi(1-\xi) + O\left(\frac{1}{N^3}\right) + \cdots\right]^{1/2}. \tag{20}$$

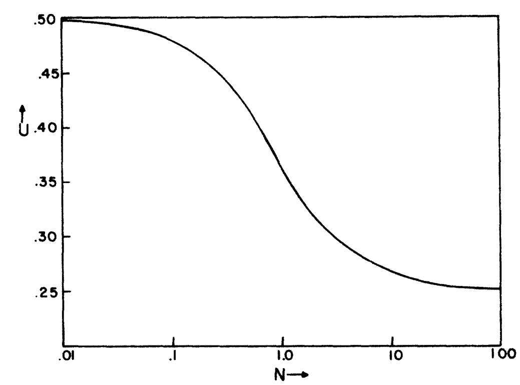

FIG. 3. The dependence of the uncertainty product U [defined in Eq. (8)] on $(\Delta N)^2 = N$ is shown for the coherent states. Note that U is independent of ξ.

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[1]See, e.g., W. Heitler, Quantum Theory of Radiation (Oxford University Press, London, 1954), Chap. 2.

[2]L. Susskind and J. Glowgower, Physics 1, 49 (1964).

[3]W. H. Louisell, Phys. Letters 7, 60 (1963).

[4]R. J. Glauber, Phys. Rev. 131, 2766 (1963).

[5]E. C. G. Sudarshan, Phys. Rev. Letters 10, 277 (1963).

[6]The operators S and C have a continuous eigenvalue spectrum in the interval -1 to $+1$. The eigenvalue can be labeled by $\sin\varphi$ and $\cos\varphi$, respectively, for S and C; however, the sum of the squares does not equal unity except in the limit of large N. In the classical limit the quantity φ corresponds to the classical phase angle. Although S^2+C^2 is not equal to the unit operator [cf. Eq. (13)] the quantities S and C bear such a close relation to the classical quantities that we have retained a notation suggestive of this correspondence.

[7]R. J. Glauber, Phys. Rev. 84, 395 (1951).

[8]W. Thirring, Principles of Quantum Electrodynamics (Academic Press, Inc., New York, 1958), Chap. 9.

[9]Another useful form is $|\alpha\rangle = A(\alpha)|0\rangle$, where $|0\rangle$ is the ground state and $A(\alpha)$ is the unitary operator $\exp(\alpha a^* - \alpha^* a)$. The $A(\alpha)$ give a non-Abelian ray representation of the Abelian group of phase translations in the variable α of the coherent states $|\alpha\rangle$, where α ranges over the complex α plane. The multiplication law is

$$A(\alpha_2)A(\alpha_1) = \exp[\tfrac{1}{2}(\alpha_2\alpha_1^* - \alpha_1\alpha_2^*)]A(\alpha_2+\alpha_1).$$

[10]P. Carruthers and M. M. Nieto (to be published) discuss the problem of the forced quantum oscillator from this point of view.

[11]In calculating (15) the fact that

$$1/(n+1)^{z+1} = \int_0^\infty t^z e^{-(n+1)t}dt/\Gamma(z+1)$$

is useful. Here $z = -\frac{1}{2}$.

[12]H. Brunet, Phys. Letters 10, 172 (1964).