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GOLDSTONE THEOREM IN NONRELATIVISTIC THEORIES

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A theorem of Goldstone,[1] when considered in its nonrelativistic form, seems to show that a many-particle system such as a superconductor, condensed Bose gas, or ferromagnet, with a ground state of lower symmetry then the Hamiltonian, must have an excitation mode with energy tending to zero for long wavelength. The apparent absence of such a mode in real superconductors and the theoretical work of Anderson,[2] which showed that it is obliterated by the finite-energy plasma oscillation in superconductors, has led to considerable theoretical interest in how and under what conditions the theorem can fail to apply. Looking in detail at a suggestion made by Guralnik, Hagen, and Kibble[3] in related relativistic work, we find that the theorem is irrelevant in the following sense. Whenever there is a finite-energy excitation mode at long wavelength, the existence of this mode can invalidate a crucial step in the theorem, and it is therefore completely consistent with the theorem for no mode with zero energy at long wavelength to be found as long as there is one of finite energy to take its place.

We first sketch the nonrelativistic form of the theorem proof,[4] and point out the crucial step. To be specific, we will refer to superconductivity whenever possible. If $n(xt)$ and $\vec{j}(xt)$ are the particle- and current-density operators, respectively, then for any operator, $\varphi_1(xt)$, the microscopic particle-conservation law requires

$$\frac{\partial}{\partial t}\langle[n(xt), \varphi_1(x't')]\rangle + \nabla\cdot\langle[\vec{j}(xt), \varphi_1(x't')]\rangle = 0. \quad (1)$$

For the superconductor it is simplest to use for $\varphi_1(xt)$ the pair creation operator,

$$\varphi_1(xt) = \psi_+^\dagger(xt)\psi_-^\dagger(xt) \quad (2)$$

with the equal time-commutation relation

$$[n(xt), \varphi_1(x't)] = 2\varphi_1(x't)\delta^3(x-x'). \quad (3)$$

The condition that the ground state be degenerate and of lower symmetry is that the ground-state expectation value, $\langle\varphi_1(xt)\rangle$, is nonzero and equal to φ even though the Hamiltonian commutes with the total-number operator.

The theorem continues with an integration of Eq. (1) over a large volume, V:

$$\int_V d^3x\, \nabla\cdot\langle[\vec{j}(xt), \varphi_1(x't')]\rangle = -\int_V d^3x \frac{\partial}{\partial t}\langle[n(xt), \varphi_1(x't')]\rangle. \quad (4)$$

As the volume tends to infinity, the right-hand side can be calculated from the spatial Fourier transform $L(k, t-t')$, where

$$L(k, t-t') = \int d^3x\, e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}\langle[n(xt), \varphi_1(x't')]\rangle. \quad (5)$$

Also applying the divergence theorem, we get

$$\int_V d\vec{S}\cdot\langle[\vec{j}(xt), \varphi_1(x't')]\rangle = -\frac{\partial}{\partial t}\lim_{k\to 0} L(k, t-t'). \quad (6)$$

If the surface integral is zero, then $\lim_{k\to 0} L(k, t-t')$ must be independent of time; and therefore in terms of $L(k\omega)$, where

$$L(k\omega) = \int dt\, e^{i\omega t} L(k, t), \qquad (7)$$

we get the requirement that

$$\lim_{k\to 0} L(k\omega) = 2\pi C_2 \delta(\omega), \qquad (8)$$

with the constant C_2 to be determined from a sum rule derived from Eq. (3).

From the definition of $L(k\omega)$ we see that

$$\int \frac{d\omega}{2\pi} \lim_{k\to 0} L(k\omega) = \int d^3x \langle [n(xt), \varphi_1(x't)] \rangle, \qquad (9)$$

and therefore from Eqs. (3) and (8) we get

$$C_2 = 2\varphi \neq 0. \qquad (10)$$

By writing the commutator $L(k\omega)$ in terms of intermediate states, one can show that in the limiting process of Eq. (8) the δ function is in fact coming from $\lim_{k\to 0}\delta[\omega-\epsilon(k)]$, where $\epsilon(k)$ is the dispersion for some excitation branch, and therefore for this branch $\lim_{k\to 0}\epsilon(k)=0$. Since we calculate in the limit as $k\to 0$ and not at $k=0$, we do not couple to other ground states as intermediate states, and therefore the $\delta(\omega)$ is not a reflection of the trivial and dynamically uninteresting zero energy required to change the system from one ground state to another.[5]

The crucial step to which we call attention is that of dropping the surface integral in Eq. (6). If there is an excitation branch such as the plasma oscillation such that

$$\lim_{k\to 0} \epsilon(k) = \omega_p \neq 0, \qquad (11)$$

then such a surface integral can be nonzero and time dependent, oscillating at the frequency ω_p. This relaxes the very restrictive form of Eq. (8) and replaces it by

$$\lim_{k\to 0} L(k\omega) = 2\pi C_2 \delta(\omega) + \pi C_1 [\delta(\omega-\omega_p) + \delta(\omega+\omega_p)] + \pi C_3 [\delta(\omega-\omega_p) - \delta(\omega+\omega_p)]. \qquad (12)$$

When the remaining steps of the proof are carried out, we get

$$2\varphi = C_2 + C_1, \qquad (13)$$

and the condition $\varphi \neq 0$ can be satisfied even with $C_2 = 0$.

It now becomes a dynamical question whether or not C_2 is in fact nonzero since it no longer follows just from the microscopic conservation law and the lower symmetry condition. It was this dynamical question which Anderson[2] solved at zero temperature for the superconductor, and his work, from this point of view, shows C_2 is zero. With C_2 zero, the representation of the commutator in terms of intermediate states contains no contribution from an excitation branch with zero energy at long wavelength.

Klein and Lee[4] incorrectly assumed that the surface integral was zero and invoked sum-rule arguments for support. But just in those cases where the Goldstone theorem is inapplicable these sum-rule arguments fail, since the microscopic conservation law is not trivially related to a macroscopic conservation law due to the very surface terms they wish to discard. Furthermore, we can now see how the theorem can make a smooth and simply understandable transition from applicability to inapplicability. If $C_2 = 0$ and $C_1 = 2\varphi$ for the case where $\omega_p > 0$, then as the interactions in the theory are changed so that $\omega_p \to 0$, we go smoothly over to the situation in which the right-hand side of Eq. (12) has the form $2\varphi 2\pi\delta(\omega)$. The constant C_2 need not arise in some discontinuous way dependent on the form of the interaction, as is required by other explanations of the theorem's inapplicability.[4,6]

We now look in more detail at the type of surface integral which appears in Eq. (6). For the electron gas with Coulomb interactions at zero temperature, and for $kk_F/m \ll \omega$,

$$\int d^3x\, e^{-i\vec{k}\cdot\vec{x}} \int dt\, e^{i\omega t} \langle [n(xt), \vec{j}(00)] \rangle = \frac{\pi m}{m} \vec{k} [\delta(\omega-\omega_p) + \delta(\omega+\omega_p)]. \qquad (14)$$

Here ω_p is the plasma frequency and n the electron density. For the moment let us consider that $\vec{j}(xt)$ can be written in a particularly suggestive form which we will elaborate later,

$$\vec{j}(xt) = i \frac{m}{2m\varphi'} \nabla \varphi_1'(xt). \qquad (15)$$

If

$$L'(k\omega) = \int d^3x\, e^{-i\vec{k}\cdot\vec{x}} \int dt\, e^{i\omega t} \langle [n(xt), \varphi_1'(00)] \rangle, \qquad (16)$$

then for $kk_F/m \ll \omega$, Eqs. (14) and (15) yield

$$L'(k\omega) = 2\varphi'\pi [\delta(\omega-\omega_p) + \delta(\omega+\omega_p)]. \qquad (17)$$

In $L'(k\omega)$, we have a candidate for $L(k\omega)$, defined earlier, such that in the zero-k limit $L(k\omega)$ is neither zero nor proportional to $\delta(\omega)$ regardless of the microscopic conservation law obeyed by $n(xt)$.

It is a simple matter to show explicitly that this residual time dependence embodied in the form of the right-hand side of Eq. (17) does in fact come from a surface term which has a simple physical interpretation. From the definition of $L'(k\omega)$ and the microscopic conservation law, it follows that

$$\int dt\, e^{i\omega t}\int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\langle[\vec{j}(xt), \varphi_1'(00)]\rangle = \frac{\omega\vec{k}}{k}L'(k\omega), \tag{18}$$

and therefore if $\Sigma(t)$ is defined by

$$\Sigma(t) = \int d\vec{S}\cdot\langle[\vec{j}(xt), \varphi_1'(00)]\rangle, \tag{19}$$

then

$$\Sigma(t) = \int\frac{d\omega}{2\pi}e^{-i\omega t}\int\frac{d^3k}{(2\pi)^3}\int d\vec{S}\cdot\vec{k}\, e^{i\vec{k}\cdot\vec{x}}\left(\frac{\omega L(k\omega)}{k^2}\right). \tag{20}$$

If we choose as our surface a large sphere centered at the origin and with radius R, the surface integral can be done exactly:

$$\Sigma(t) = \int\frac{d\omega}{2\pi}e^{-i\omega t}\int\frac{d^3k}{(2\pi)^3}\omega L'(k\omega)2i\pi^2R \times\left\{\frac{(kR)^{-1}\sin kR - 2\cos kR}{\pi k^2 R}\right\}, \tag{21}$$

The bracketed expression is a representation of $\delta(|k|)$ as R goes to infinity as long as $L'(k\omega)$ behaves well enough for large momenta. Since this will be the case, we can replace $L'(k\omega)$ by $L'(0\omega)a^2/(k^2+a^2)$ where this last factor just simulates L' going to zero at large momentum. The final answer is independent of a^2 as $R\to\infty$; this factor is only one of many cutoff mechanisms which could be used. In spherical polar coordinates, Eq. (21) is written

$$\Sigma(t) = \int\frac{d\omega}{2\pi}e^{-i\omega t}i\omega L'(0,\omega) \times\frac{2}{\pi}\int_0^\infty dk\frac{a^2R[(kR)^{-1}\sin kR - \cos kR]}{k^2+a^2}, \tag{22}$$

and after the final integration,

$$\lim_{R\to\infty}\Sigma(t) = \int\frac{d\omega}{2\pi}e^{-i\omega t}i\omega L'(0,\omega). \tag{23}$$

Relating this to Eq. (16), we see precisely that

$$\lim_{R\to\infty}\Sigma(t) = -\frac{\partial}{\partial t}\lim_{k\to 0}\int d^3x\, e^{-i\vec{k}\cdot\vec{x}} \times\langle[n(xt), \varphi_1'(00)]\rangle, \tag{24}$$

which is to be compared with Eq. (4).

The physical meaning of the surface term is quite clear. If we consider a finite volume in an infinite electron gas and there is a plasma oscillation of wavelength longer than the linear dimension of the volume, then there will be an oscillating total charge within the volume and corresponding current across its surface. If we let the considered volume increase, keeping the excitation wavelength longer than the linear dimension, then, as long as the excitation frequency remains finite, the total charge as well as current crossing the surface will continue to oscillate in time. In the calculation just completed the mathematical representation of this did occur. As R increases, the δ-function form leaves in the final frequency integral only frequencies corresponding to excitations of wavelength comparable to or larger than R and, finally, in the limit projects out the $k=0$ contribution.

We must now indicate the relevance of Eq. (15) and the relationship of the operators $\varphi_1'(xt)$ and $\varphi_1(xt)$. In any of the theories of the type we are considering, a new parameter is found which designates the various degenerate ground states. In ferromagnetism the parameter is an angle designating magnetization direction, while in the Bose gas and superconductor it is the phase of the condensate and electron-pair wave function, respectively. The excitation modes we are concerned with are excited when the system is forced into a configuration very similar to a ground state, except that this parameter varies slowly in space. In ferromagnetism, for example, by an external field we might force the magnetization direction to vary slowly in space. If we suddenly turn off the external field, the system will begin to oscillate with some characteristic frequency or frequencies. The Goldstone theorem then deals with whether or not such frequencies must tend to zero as the characteristic length in the original distortion tends to infinity. It is clear in the ferromagnetic case as well as the plasma example given above that if one looks only within a volume smaller than the distortion length, and if the frequency of oscillation is finite, magnetic moment must flow in and out of the region. In fact, if the ferromagnetic aligning forces are of sufficiently long range, the spin-wave energies will not tend to zero for infinite wavelength, and, just as in the superconductor, a surface-term contribution will lead to a consistent condition of inapplicability of the Goldstone theorem.

In the superconductor and condensed Bose gas, the variation of the wave-function phase from place to place is associated with a current. If spatial variation is found only in the phase, we may write for the superconductor

$$\langle\varphi_1(xt)\rangle = \langle\psi_+^\dagger(xt)\psi_-^\dagger(xt)\rangle = e^{-i\alpha(x)}\langle\psi_+^\dagger\psi_-^\dagger\rangle. \quad (25)$$

Considering $\langle\varphi_1(xt)\rangle$ as a wave function, we may associate $\nabla\alpha(x)$ with a local momentum and $\nabla\alpha(x)/2m$ with a superfluid velocity, having divided by $2m$, the mass of the pair referred to in the wave function. If this velocity is multiplied by the density of superconducting electrons, which at zero temperature is the total density, a current results. Thus, in terms of expectation values, we can write

$$\langle\vec{j}(xt)\rangle = \frac{m}{2m} i \frac{\nabla\langle\varphi_1(xt)\rangle}{\langle\varphi_1(xt)\rangle}. \quad (26)$$

The fact that Eq. (26) is an equation of expectation values and not the operator relationship, Eq. (15), in no way weakens the point of this paper. We have shown that the operator on the right-hand side of Eq. (15) with $\varphi_1'(xt)$ replaced by $\varphi_1(xt)$ is certainly closely related to a current operator. Therefore, the physical and mathematical arguments following from charge conservation which we can perhaps more easily grasp in terms of the normal electron gas or ferromagnet are related to superconductivity in the manner suggested by Eqs. (15), (16), and (17). $L'(k\omega)$ is not just a candidate for $L(k\omega)$, but has the form we would expect for $L(k\omega)$ when calculated at zero temperature in the theory of superconductivity.

In conclusion, the Goldstone theorem fails to determine that the energy of an excitation branch tends to zero for infinite wavelength because of the nature of macroscopic conservation laws and related surface integrals when finite-energy, long-wavelength modes exist. The theorem is irrelevant in nonrelativistic theories, since one must always do a dynamical analysis, strongly dependent on the range of forces in the problem, to determine the nature of the modes; no definite statement follows just from the microscopic conservation law and lower symmetry condition.

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[1]J. Goldstone, Nuovo Cimento 19, 154 (1961).

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[3]G. Guralnik, C. R. Hagen, and T. W. B. Kibble, Phys. Rev. Letters 13, 585 (1964).

[4]A. Klein and B. W. Lee, Phys. Rev. Letters 12, 266 (1964).

[5]Such coupling was claimed responsible for the theorem's failure by Klein and Lee instead of the nonzero surface term discussed in the present paper.

[6]The reason for the theorem's failure suggested by Walter Gilbert [Phys. Rev. Letters 12, 713 (1964)], which we suggest is not relevant, differs markedly from our result in the nature of this transition.

[7]The relationship between the gradient of the phase of the anomalous matrix element and a superfluid velocity has been discussed by many authors. In the condensed Bose gas it has been well expounded in the work of L. P. Gross.