13, 173 (1964); B. Sakita, Phys. Rev. 136, B1756 (1964). See also A. Pais, Phys. Rev. Letters 13, 175 (1964).

[2]K. Bardakci, J. M. Cornwall, P. G. O. Freund, and B. W. Lee, Phys. Rev. Letters 13, 698 (1964); 14, 48 (1965).

[3]R. P. Feynman, M. Gell-Mann, and G. Zweig, Phys. Rev. Letters 13, 678 (1964). See also S. Okubo and R. Marshak, Phys. Rev. Letters 13, 818 (1964).

[4]F. Gürsey, A. Pais, and L. A. Radicati, Phys. Rev. Letters 13, 299 (1964).

[5]M. A. B. Bég, B. W. Lee, and A. Pais, Phys. Rev. Letters 13, 514 (1964); B. Sakita, Phys. Rev. Letters 13, 643 (1964).

[6]M. Gell-Mann, Phys. Rev. Letters 14, 77 (1965). Couplings of the $L=2,4,\cdots$ Regge recurrences of the meson 35-plet can be discussed along the lines of the present paper. Of course these couplings will be of the derivative type.

[7]The matrix $M$ for low-energy mesons has been first written down by M. A. B. Bég and A. Pais, Phys. Rev. (to be published). Our derivation is not restricted to small meson momenta. For the trilinear and quadrilinear meson couplings (as opposed to meson-baryon couplings) no static limit can be envisaged, and a momentum-independent derivation is crucial to our argument.

[8]Note that this situation is essentially different from the case of a "gauge theory" [C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954); J. J. Sakurai, Ann. Phys. (N.Y.) 11, 1 (1960)], where vector mesons couple to the conserved total unitary-spin current. This is not surprising since gauge theories are based on the kinetic-energy term of the Lagrangian whereas it is exactly this term that breaks U(6). The fact that the trilinear meson couplings identically vanish in the limit of U(6) symmetry is a particular case of a general result due to Sakurai (to be published) stating that this must be the case in any theory in which the nine vector mesons are degenerate. One of the authors (B.W.L.) is grateful to Professor J. J. Sakurai for reminding him of this result.

[9]This can most easily be seen by replacing one of the matrices $M$ in (10) by $K$ and observing that Tr $MK\gamma_5 \equiv$ Tr $MK \equiv 0$. The meaning of the vertex (12) can be better understood at a pictorial level. Consider an arbitrary Feynman graph with three external meson lines. In the limit of "exact U(6)" in which symmetry-breaking kinetic-energy effects are ignored ($\gamma p$ terms in quark propagators and $k_\mu k_\nu$ terms in meson propagators are dropped), this vertex would be ~Tr $MMM$ $\equiv 0$. Considering the kinetic energy as a spurion can be visualized by kineton emission from the three-meson vertex. It is readily checked that any graph with three external meson lines and one external kineton line leads to a contribution of the form (12) to the vertex.

[10]H. J. Lipkin, Phys. Rev. Letters 13, 590 (1964).

[11]M. Gell-Mann, D. Sharp, and W. Wagner, Phys. Rev. Letters 8, 261 (1962).

[12]The tentative value $m = 615$ MeV has been suggested by M. A. B. Bég and V. Singh, Phys. Rev. Letters 13, 681 (1964).

[13]A. H. Rosenfeld et al., Rev. Mod. Phys. 36, 977 (1964).

[14]The result of reference 11 is $\Gamma(\omega) = 0.4$ MeV. For the effects of $\omega$-$\varphi$ mixing, see also R. Dashen and D. Sharp, Phys. Rev. 133, B1518 (1964); and R. Socolow, to be published.

[15]In the estimate of Eq. (21), the contribution from the diagram $K^* \to \rho + K$ is also included. The relativistic correction factor $W(m_{K^*})$ of reference 11 is taken to be $\simeq 2$.

[16]See, e.g. J. J. Sakurai, Phys. Rev. 132, 434 (1963).

[17]J. J. Sakurai, Rendiconti della Scuola Internazionale di Fisica "Enrico Fermi" XXVI Corso, p. 42. In particular, the agreement with experiment of the relation between $g_{\rho\pi\pi}$ and $g_{\pi N}$ obtained in reference 4 assuming both universality in the usual sense $[(g_\rho^2/4\pi)_{\rm univ} \cong 2]$ and U(6) symmetry is presumably fortuitous.

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# RELATIVISTIC, CROSSING-SYMMETRIC, SU(6)-INVARIANT $S$-MATRIX THEORY

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In recent papers[1,2] we have described several aspects of the relativistic completion of SU(6). The procedure is meant to apply only to effective vertices and matrix elements ("$S$-matrix theory"). For local Lagrangian field theory there is a specific breakdown[1,2] of SU(6). In this Letter we indicate how the systematic application of our rules leads to an $S$-matrix description which is fully covariant, crossing symmetric, and SU(6) invariant. As before,[1,2] we take as our starting definition for SU(6) a group property of zero three-momentum one-particle states.

It is crucial to note that the relativistic completion (boosting[3]) of an SU(6) representation is not unique, except for the trivial case[4] of the 6, which of course is to be treated like the representation $[\frac{1}{2}, 0]$ of $\mathfrak{L}^{\uparrow}$, the orthochronous Lorentz group including space reflections. Thus we call it $\underline{6}[\frac{1}{2}, 0]$. This is given by $u^{ia}(p)t^A$ where the notation is as follows: $i=1,2$ denotes spin states, $A=1,2,3$ refers to unitary spin,

$a=1,2$ denotes the upper (lower) component pair of the four-spinor $u$. Thus if we have a set of (Hermitian) matrices $\gamma_\lambda$, we may exemplify our notation in the useful representation[5] $\vec{\gamma}=-\rho_2\vec{\sigma}$, $\gamma_4=\rho_1$, $\gamma_5=\rho_3$ as follows: $(\gamma_1 u)^{ia}$ is short for $(-\rho_2)_b{}^a(\sigma_1)_j{}^i u^{jb}$, etc. Thus for fixed $i$, $u^{ia}$ is a two-component spinor in "$\rho$ space." With $\bar{u}=u^\dagger\gamma_4$ we have $(\bar{u}\rho_3)_{ia}=\bar{u}_{jb}(\rho_3)_a{}^b \times\delta_i{}^j$, etc. We can briefly denote the particle states of $\underline{6}[\frac{1}{2},0]$ by $u^\lambda(p)$, $\lambda=(i,a,A)=1,\cdots,12$. We reserve the early letters $a,b,\cdots$ for $\rho$ space; $i,j,k,\cdots$ for spin states. We denote the antiparticle states by $v^\lambda(p)$. The Hermitian adjoint functions are $u_\lambda{}^\dagger, v_\lambda{}^\dagger$.

For the $\underline{35}$ there are two distinct completions (for the same set of physical mesons!). They are $\underline{35}[\frac{1}{2},\frac{1}{2}]^-$ and[5] $\underline{35}[(\frac{1}{2})^2,0]^-$, to which there will correspond respective $12\times12$ meson matrices ${}^{(1)}\mathfrak{M}_\mu{}^\lambda$ and ${}^{(2)}\mathfrak{M}_\mu{}^\lambda$. We have,[1] for $35[\frac{1}{2},\frac{1}{2}]^-$,

$$^{(1)}\mathfrak{M}_\mu{}^\lambda(q)=\left[-\gamma_5\frac{(\gamma q)}{\mu_{00}}\right]_{jb}^{ia}P_B{}^A+[i(\gamma\epsilon)]_{jb}^{ia}V_B{}^A, \quad (1)$$

where $P$ is the pseudoscalar octet, $V$ the vector nonet; $\gamma q=\gamma_\lambda q_\lambda=\gamma q+i\gamma_4 q_0$. Furthermore,[2] for $35[(\frac{1}{2})^2,0]^-$,

$$^{(2)}\mathfrak{M}_\mu{}^\lambda(q)=i\left[(\gamma_5)_b{}^a\delta_j{}^iP_B{}^A+\left(\frac{\sigma_{\mu\nu}q_\mu\epsilon_\nu}{\mu_{00}}\right)_{jb}^{ia}V_B{}^A\right]. \quad (2)$$

Equation (2) corresponds[6] to a reducible $\mathfrak{L}^\dagger$ representation, corresponding to a ps(ps)+v(t) structure with relative weight determined by SU(6). One may wish to reject Eq. (2) by the criterion of $\mathfrak{L}^\dagger$-irreducibility. However, we shall see below that this may not be prudent.

We note that $\bar{u}_\lambda(p_1)\,{}^{(i)}\mathfrak{M}_\mu{}^\lambda(q)u^\mu(p_2)$, $i=1,2$, $q=p_2-p_1$, are fully covariant SU(6)-invariant ($\underline{6}^*, \underline{6}, \underline{35}$) vertices for (particle, particle, meson). In order to go to a crossed channel, replace $u^\mu(p_2)$ by $v^\mu(-p_2)$, keep the rest as is, but read now $q=-p_2-p_1$. Thus we have manifest crossing symmetry. All these covariance properties are readily shown to hold likewise for any $S$-matrix element involving arbitrary numbers of $\underline{6}$'s and $\underline{35}$'s.

The $\underline{56}$. For $\vec{p}=0$ we have[7]

$$B^{\alpha\beta\gamma}=\chi^{ijk}d^{ABC}+(3\sqrt{2})^{-1}\times[\epsilon^{ij}\chi^k X^{ABC}+\epsilon^{jk}\chi^i X^{BCA}+\epsilon^{ki}\chi^j X^{CAB}],$$
$$X^{ABC}=\epsilon^{ABD}b_D{}^C. \quad (3)$$

There are again two distinct boosts, namely $\underline{56}[(\frac{1}{2})^3,0]$ and $\underline{56}[(\frac{1}{2})^2,\frac{1}{2}]$. [The set of distinct boosts for any SU(6) representation is of course fully determined by its spin content.] In order to give a compact specification of these boosts, we introduce the boost matrices $D^{(a)}$, $a=1,2$, where

$$D^{(1)}(p)=\tfrac{1}{2}(1+\gamma_5)[m-i(\gamma p)\gamma_4][2m(E+m)]^{-1/2}, \quad (4)$$

while $D^{(2)}$ is the same expression but with $(1+\gamma_5)\to(1-\gamma_5)$. Next we introduce the following definition[8]:

$$B^{\alpha a,\beta b,\gamma c}(p)=D_l{}^{(a)i}(p)D_m{}^{(b)j}(p)D_n{}^{(c)k}(p)B^{lA,mB,nC}. \quad (5)$$

We have, for $\underline{56}[(\frac{1}{2})^3,0]$, the elements

$$\begin{pmatrix}B^{\alpha1,\beta1,\gamma1}\\ B^{\alpha2,\beta2,\gamma2}\end{pmatrix}, \quad (6)$$

which is 4- (8-)component for the octet (decuplet). This is the relativistic SU(6) representation which was constructed earlier.[1] Observe that this completion maintains the $(\alpha,\beta,\gamma)$ symmetry of the $\underline{56}$ for $\vec{p}=0$. Equation (6) gives the particle states. The antiparticle states are obtained by the substitution

$$D^{(1)}(p)\to D^{(1)}(p);\; D^{(2)}(p)\to -D^{(2)}(p) \quad (7)$$

in Eq. (5). Furthermore, $\underline{56}[(\frac{1}{2})^2,\frac{1}{2}]$ is described by

Eq. (5) with $a$, $b$, and $c$ not all identical. (8)

Also here a pairing like in Eq. (6) is always necessary. This pairing is a "1-2 conjugation." Thus, for example,

$$\begin{pmatrix}B^{\alpha1,\beta1,\gamma2}\\ B^{\alpha2,\beta2,\gamma1}\end{pmatrix} \quad (9)$$

has the appropriate behavior under space reflections. Clearly, this relativistic representation no longer has the $(\alpha,\beta,\gamma)$ symmetry of SU(6).

We now turn to the baryon-meson vertex. To begin with we discuss the particle-particle sector. Consider the expressions

$$B^{\dagger}(p_1)_{\alpha a,\beta b,\gamma c}\,{}^{(1)}\mathfrak{M}_{\delta d}^{\;\gamma c}(q)B(p_2)^{\alpha a,\beta b,\delta d}, \quad (10)$$

$$B^{\dagger}(p_1)_{\alpha a,\beta b,\gamma c}\,{}^{(2)}\mathfrak{M}_{\delta d}^{\;\gamma c}(q)B(p_2)^{\alpha a,\beta b,\delta d}. \quad (11)$$

In Eqs. (10) and (11) we sum over the Greek indices [SU(6) labels]. In regard to the Latin indices we observe the pairings (with equal weight) prescribed by Eqs. (6) and (9), but do not otherwise sum over $a$, $b$, $c$, and $d$. Each of Eqs. (10) and (11) then lead to four generally distinct vertices. Each of these eight expressions represents a Lorentz-covariant SU(6)-invariant particle-particle vertex. Moreover, by application of Eq. (7), it is readily seen that each of these eight expressions is also crossing symmetric.

Thus we have now found an eightfold answer to the construction of the vertex with all the requisite symmetry properties. We have found some general properties of this set, namely: (i) Each of them reproduces the ratio[7] $\mu(n)/\mu(p)=-\frac{2}{3}$ if we replace $\mathfrak{M}$ by the SU(6)-covariant magnetic-moment operator. (ii) There are three possible values for the $D/F$ ratio of pseudoscalar mesons, namely: pure $F$, $D/F=\frac{3}{2}$, $D/F=6$. Thus a general combination of the eight vertices renders this ratio arbitrary. There exists a clear need for more selective criteria.

An elegant criterion is provided by the requirement that the vertex be invariant under the transformations of a group $SU(12)_{\mathcal{L}}$, the "booster group" of SU(6). The tensors of this group are defined so as to preserve space-reflection invariance. The $\mathcal{L}$ indicates that the algebraic structure is formally that of SU(12), provided the adjoint tensors[7] are defined in terms of the density appropriate to the Lorentz metric. Within the context of the present paper these mathematical artifices are fully justifiable.

The $\underline{56}$ can be embedded in the totally symmetric SU(12) representation $\underline{364}$. This corresponds to a unique vertex, namely

$$B_{\lambda\mu\rho}^{\;\;\dagger}(p_1)({}^{(1)}\mathfrak{M}+{}^{(2)}\mathfrak{M})_{\nu}^{\;\rho}(q)B^{\lambda\mu\nu}(p_2), \quad (12)$$

with $\lambda=\alpha a$, $\mu=\beta b$, $\rho=\gamma c$, and $\nu=\delta d$. There is a second choice for embedding the $\underline{56}$, namely in the mixed-symmetry representation $\underline{572}$. This choice gives two independent couplings. In the rest of this note we confine ourselves to the vertex given in Eq. (12). Before mentioning some implications of this coupling, it may be well to state a few general properties of this group $SU(12)_{\mathcal{L}}$.

(a) The representation $\underline{364}$ has the $SU(6)\otimes SU(2)$ content $(\underline{56},\underline{4})\oplus(\underline{70},\underline{2})$. Here SU(2) corresponds to "$D$ spin," the fundamental representation being $\underline{2}$ corresponding to the pair $(D^{(1)},D^{(2)})$. "$D$ spin" is a kinematic concept. Thus $(\underline{56},\underline{4})$ corresponds to total symmetry of boosting, that is, $D$ spin $\frac{3}{2}$. $(\underline{56},\underline{4})$ corresponds of course to one and only one $\underline{56}$ in the rest frame. Parity is still extraneous to the booster group. If $SU(12)_{\mathcal{L}}$ is now an approximate symmetry of strong interactions, a $\underline{70}$ should exist with positive parity. Note that the representation $\underline{220}$ contains one $\underline{20}$ and one $\underline{70}$, again with equal parity!

(b) Similarly, the $\underline{35}$ belongs to the booster representation $\underline{143}=(\underline{35},\underline{3}+\underline{1})\oplus(\underline{1},\underline{3})$. Here $(\underline{35},\underline{3}+\underline{1})$ corresponds to the boosts ${}^{(1)}\mathfrak{M}$ and ${}^{(2)}\mathfrak{M}$ with prescribed relative weight as it appears in Eq. (12). Clearly an $SU(12)_{\mathcal{L}}$ representation contains prescribed mixtures of $\mathcal{L}^{\dagger}$ representations corresponding to the same physical SU(6) representations in the rest frame. As an approximate symmetry, $SU(12)_{\mathcal{L}}$ predicts a further pseudoscalar singlet (boosted with $D$ spin 1). The $\eta'$ meson at 960 MeV would fit nicely in this spot.[9]

(c) We now return to the meson-baryon vertex. Equation (12) gives a unique $D$:$F$ ratio in the coupling of pseudoscalar mesons. Remarkably enough, this ratio turns out to be precisely 3:2.

(d) Equation (12) determines the Pauli coupling of the vector mesons, to the baryons, in terms of the Dirac coupling. If one assumes that the Pauli couplings of the $\rho$ and $\omega$ determine the Pauli coupling of the photon, Eq. (12) may be said to predict the magnetic moment of the proton. Before a definite value can be assigned, it is necessary to have an unambiguous procedure for handling central masses, which enter in an essential way.

The extension of the rules for constructing relativistically invariant SU(6)-invariant structures given above for the three-point functions can readily be extended to $n$-point functions. $SU(12)_{\mathcal{L}}$ requirements will again reduce the number of independent form factors. Explicit constructions will be given elsewhere. We only note here that the formalism incorporates the relation found by Johnson and Treiman[10] in a Lorentz-invariant framework.

The present considerations do not affect the conclusion reached earlier[1] that local Lagrangian field theory and SU(6) invariance are not compatible (except in the trivial case that there is no interaction at all).

The notion of $SU(12)_{\mathfrak{L}}$ invariance brings up mathematical problems of a novel nature. These will be studied further.

After the conclusion of this work one of us (A.P.) was informed in a private communication by Professor Abdus Salam that he had obtained full $S$-matrix covariance along similar lines.

Note added in proof.—In different contexts, the group $SU(12)_{\mathfrak{L}}$ appears in earlier papers.[11]

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[1]M. A. B. Bég and A. Pais, "Lorentz Invariance and the Interpretation of SU(6). I," to be published.

[2]M. A. B. Bég and A. Pais, "Lorentz Invariance and the Interpretation of SU(6). II," to be published.

[3]The use of this term is discussed by S. Weinberg, Phys. Rev. 133, B1318 (1964), especially Sec. 2.

[4]The 6 is the only representation of SU(6) for which spin and unitary spin factorize. For this reason, models which involve sextets only hide many of the complexities of the problem.

[5]This representation is useful for demonstrating invariance under $L_+^\uparrow$. With appropriate labelings, our methods hold for any representation.

[6]Here $(\frac{1}{2})^2 \equiv \frac{1}{2}\otimes\frac{1}{2} = 1 \oplus 0$. Thus $[(\frac{1}{2})^2, 0]$ is a convenient shorthand for the reducible representation $[1,0]\oplus[0,0]$.

[7]We follow the notations given in M. A. B. Bég, B. W. Lee, and A. Pais, Phys. Rev. Letters 13, 514 (1964); see especially footnote 9 of that paper.

[8]Note that in our notation, $\chi_{(i,a)}^\dagger$ transforms according to $[D^{(a)}]^{-1}$.

[9]M. Goldberg et al., Phys. Rev. Letters 12, 546 (1964).

[10]K. Johnson and S. B. Treiman, Phys. Rev. Letters 14, 189 (1965).

[11]P. Roman and J. J. Aghassi, Phys. Letters 14, 68 (1965) [SV(12); with a different physical interpretation of multiplets]. K. Bardakci, J. M. Cornwall, P. G. O. Freund, and B. W. Lee, Phys. Rev. Letters 14, 48 (1965) [$M(12)$; encountered in the discussion of W(6)].

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# NONEQUIVALENCE OF THE ONE-CHANNEL $N/D$ EQUATIONS WITH INELASTIC UNITARITY AND THE MULTICHANNEL $ND^{-1}$ EQUATIONS

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Consider a partial-wave elastic-scattering amplitude[1] for two spinless particles of equal mass, $M$, as a function of $s = 4(k^2+M^2)$:

$$A = \frac{1}{2i\rho}(S-1) = \frac{1}{2i\rho}(\eta e^{2i\delta}-1) = B + {}^R A, \qquad (1)$$

where $\rho$ is a kinematical factor and the "generalized potential" $B$ is regular in the physical region, whereas ${}^RA$ has cuts only for $s > 4M^2 \equiv s_E$. The inelastic partial-wave cross section $\sigma_r^{\,l}$ is determined by $\eta$ alone:

$$\sigma_r^{\,l} = \pi k^2(2l+1)(1-\eta^2). \qquad (2)$$

Given $B$ and $\eta$, we can determine $A \equiv N/D$ using the Frye-Warnock equations[2,3]

$$\frac{2\eta(s)}{1+\eta(s)}\mathrm{Re}N(s) = \bar{B}(s) + \frac{1}{\pi}\int_{s_E}^{\infty}\frac{[\bar{B}(s')-\bar{B}(s)]2\rho(s')\,\mathrm{Re}N(s')ds'}{(s'-s)[1+\eta(s')]},$$

$$\bar{B}(s) = B(s) + \frac{\mathrm{P}}{\pi}\int_{s_I}^{\infty}\frac{[1-\eta(s')]ds'}{2\rho(s)(s'-s)},$$

$$D(s) = 1 - \frac{\mathrm{P}}{\pi}\int_{s_E}^{\infty}\frac{2\rho(s')\,\mathrm{Re}N(s')ds'}{(s'-s)[1+\eta(s')]} - i\frac{2\rho(s)}{1+\eta(s)}\mathrm{Re}N(s)\theta(s-s_E),$$

$$\mathrm{Im}N(s) = \frac{1-\eta(s)}{2\rho(s)}\mathrm{Re}D(s)\theta(s-s_I) \qquad (3)$$