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VISCOSITY OF TYPE-II SUPERCONDUCTORS

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When the Lorentz driving force $(\vec{J}\times\vec{\varphi}_0)/c$ on the flux quanta in a type-II superconductor carrying a current $\vec{J}$ exceeds the pinning force, a voltage appears along the superconductor. This voltage has recently been measured by Kim et al.[1] and arises by induction from the flow of vortices driven across the superconductor by the transport current. The flow of vortices is opposed by viscous forces of the form $-\eta v_L$, where η is the viscosity coefficient and v_L the velocity of the vortex. The motion of vortices has been discussed by Anderson and Kim[2] and more recently by one of us,[3] who has shown that the only force acting on a vortex line is the Lorentz force $(\vec{J}\times\vec{\varphi}_0)/c$ arising out of the transport current. The object of this note is to discuss the viscosity coefficient η.

Kim and co-workers have shown that their experimental results over a wide range of temperature and composition can be described by the empirical formula

$$\eta_{\text{emp}} = \pi\hbar H_{c2}\sigma/ec, \qquad (1)$$

where H_{c2} is the upper critical field and σ the conductivity in the normal state. They suggest that the friction may result from currents flowing in the core of the vortex line, a cylinder with radius a approximately equal to the coherence distance ξ. This model follows from a calculation of Caroli, de Gennes, and Matricon,[4] who have shown that in this central region of the vortex the energy gap is so small that the conductivity is practically normal. It is found that η is much larger than can be accounted for by eddy currents resulting from the electric field $\vec{E}_m = -(1/c)\vec{v}_L\times\vec{H}$ generated by the moving magnetic field of the vortex line, which was used as the basis of a previous calculation by Volger, Staas, and Vijfeijken.[5] We show that there is another contribution to the electric field which can be much larger in the vicinity of the core and leads to an expression for η close to the empirical one.

We adopt a simple model for the vortex such that the core of radius a is normal with conductivity σ, and outside of the core the metal is superconducting. The materials used by Kim et al. are type-II superconductors with $\xi \ll \lambda$, where λ is the penetration depth. One may expect to describe these materials approximately by a local theory in which the current density is a function of $m\vec{v}_s(r_s) = \vec{p}_s(r) + (e/c)\times\vec{A}(r)$, where $\vec{p}_s$ is the common momentum of the paired electrons in the ground state and $\vec{A}(r)$ is the vector potential. The quantum condition for unit flux,

$$2\oint \vec{p}_s\cdot d\vec{l} = h,$$

gives $p_{s\theta} = \hbar/2r$. When the vortex line is moving, $\vec{r}$ is replaced by $\vec{r}-\vec{v}_L t$; to a close approximation the current distribution is unmodified by the motion. However, an additional electric field beyond that from $\vec{A}(\vec{r}-\vec{v}_L t)$ is required to change $\vec{v}_s$ with time as the vortex line moves past. This additional field is large near the core and is responsible for the major part of the energy loss when $\xi \ll \lambda$. The equation of motion for v_s, if we keep only the terms dependent on v_L, is[6]

$$m\,\partial\vec{v}_s/\partial t = -(\vec{v}_L\cdot\vec{\nabla})\vec{v}_s = -(e/m)\vec{E}. \qquad (2)$$

The field may be expressed in the form[3]

$$\vec{E} = -(1/c)(\vec{v}_L\times\vec{H}) - \vec{\nabla}\varphi, \qquad (3)$$

where the electrostatic potential φ outside of the core is given by

$$-e\varphi = m\vec{v}_L\cdot\vec{v}_s. \qquad (4)$$

This choice of φ is consistent with electrical neutrality, $\operatorname{div}\vec{E}=0$. When $\xi \ll \lambda$, $\vec{A}$ is small compared with $\vec{p}_s$ near the core, and one may replace $m\vec{v}_s$ by $\vec{p}_s$.

If we take v_L in the x direction and require that φ be continuous across the boundary, we find

$$\varphi = -\hbar v_L y/2er^2, \quad r>a; \tag{5a}$$

$$\varphi = -\hbar v_L y/2ea^2, \quad r<a. \tag{5b}$$

The field within the core is uniform and equals

$$E_i = \hbar v_L/2ea^2. \tag{6}$$

There is a charge density $(\hbar v_L/4\pi e a^2)\sin\theta$ developed at the surface of the vortex core. In a more realistic model of a vortex without a sharp boundary, this charge would be expected to be distributed over a distance of the order of ξ. The Coulomb energy associated with the charge is very small.

The normal current density in the core, σE_i, is very small compared with the supercurrent flow just outside the core for any reasonable normal conductivity. A small additional flow outside the core is required to satisfy the continuity of current, but this will have very little effect on the flow pattern and on E_i.

The radius of the core may be estimated from the critical value of v_s for which the energy-gap parameter, Δ, goes to zero and the metal becomes normal:

$$a = \hbar/(2mv_{sc}). \tag{7}$$

When evaluated from the local theory,[7] we find that near $T=0°$K, a is close to the coherence distance ξ. For the impure case with a mean free path $l \ll \xi_0$, we take $\xi = (l\xi_0)^{1/2}$. Near T_c, where the Ginzburg-Landau theory applies,[7]

$$m^2 v_{sc}^{\;2} = 2H_L^{\;2}\lambda^2 e^2/c^2 = \tfrac{1}{2}\hbar e H_{c2}/c. \tag{8}$$

Equating the dissipation in the core per unit length of line, $\pi a^2 \sigma E^2$, to $\eta v_L^{\;2}$, we find

$$\eta = \frac{\pi\hbar^2\sigma}{4e^2a^2} = \frac{\pi m^2 v_{sc}^{\;2}\sigma}{e^2}. \tag{9}$$

When v_{sc} is expressed in terms of H_{c2}, we find an expression similar to the empirical one:

$$\eta = \alpha\pi\hbar H_{c2}\sigma/ec = \alpha\eta_{\text{emp}}, \tag{10}$$

where α is a numerical factor which varies little with temperature and impurity concentration and is equal to $\frac{1}{2}$ near T_c. Dissipation outside of the core may be expected to contribute a comparable amount, making the agreement between theory and experiment even closer.

Other than near T_c, there is doubt about the applicability of the local theory in the immediate vicinity of the core. (The boundary of the core will actually be spread out over a coherence distance.) Outside of this region, the equations we have used can be justified in terms of the microscopic theory. In the presence of fields A and φ, the induced charge and current in a superconductor at $T=0°$K are[8]

$$-4\pi\lambda_D^{\;2}\rho_{\text{ind}} = \varphi + \frac{1}{c}\frac{\partial W}{\partial t}, \tag{11}$$

$$-4\pi\lambda^2 \vec{j}_{\text{ind}} = \vec{A} - \nabla W, \tag{12}$$

where W is the phase of the energy gap defined by $\Delta = |\Delta|\exp(-2ieW/c)$ and λ_D is the Debye length. In our notation $\vec{p}_s = -(c/e)\nabla W$, and in the linear range in which these equations apply, $\vec{j}_{\text{ind}} = -ne\vec{v}_s$. The time derivative of (12) with use of (11) leads to (2). In our model, the induced charge vanishes outside of the core.

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