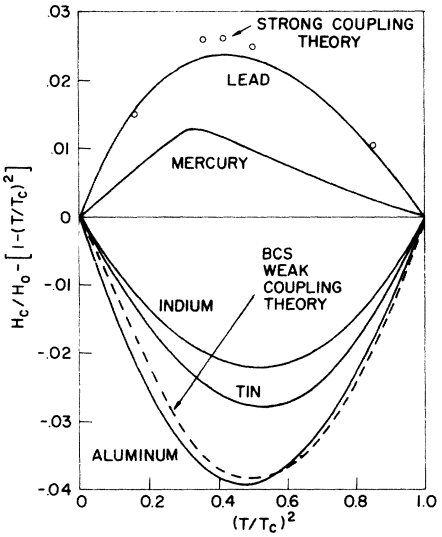

FIG. 1. Deviation of the reduced critical field from a parabolic curve as a function of the square of the reduced temperature. The experimental curves are taken from D. K. Finnemore, D. E. Mapother, and R. W. Shaw, Phys. Rev. 118, 127 (1960), and N. E. Phillips, Phys. Rev. 114, 676 (1959), while the circles are the results of our calculation using case 1 for our model of lead.

the renormalization parameter. In our model

$$m^*/m = Z_{n1}(\omega = 0, T = 0)$$

where Z_{n1} is the real part of Z for the normal state ($\Delta = 0$). The values of m^*/m obtained in this manner are listed in Table I and compared with the experimental estimate of the effective mass obtained from specific heat measurements. It is clear that a large part of the effective-mass shift can be accounted for by the electron-phonon interaction.

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LEAD PHONON SPECTRUM CALCULATED FROM SUPERCONDUCTING DENSITY OF STATES

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The present theory of superconductivity,[1-4] with the Eliashberg phonon-induced interaction between electrons, predicts structure in the electronic density of states which is related to structure in the phonon density of states. The electronic density of states is measured directly in tunneling experiments,[4-7] and this structure has been observed in lead and several other metals.[8-11] Schrieffer, Scalapino, and Wilkins[4] have computed the electronic density of states in lead by solving the energy-gap equation, taking a simple model for the phonon density of states and coupling constants, and obtain reasonably good agreement with tunneling experiments of Rowell, Anderson, and Thomas.[10] However, it is evident from Fig. 1 of reference 10 that the phonon density of states in lead must contain considerably more structure than the model considered by Schrieffer, Scalapino, and Wilkins. It is therefore of in-

terest to devise a procedure for "inverting" the gap equation to obtain a phonon density of states and coupling constants from the experimental information on the electronic density of states. We report below a procedure which accomplishes this and obtain the strength of the electron-phonon coupling, the screened Coulomb interaction, and the phonon density of states in lead.

The integral equations for the normal and pairing self-energies of a dirty superconductor[12] are

$$\xi(\omega)=[1-Z(\omega)]\omega=\int_{\Delta_0}^{\infty}d\omega'\,\mathrm{Re}\left[\frac{\omega'}{(\omega'^2-\Delta'^2)^{1/2}}\right]\times\int d\omega_q\,\alpha^2(\omega_q)F(\omega_q)[D_q(\omega'+\omega)-D_q(\omega'-\omega)], \tag{1}$$

$$\varphi(\omega)=\int_{\Delta_0}^{\omega_c}d\omega'\,\mathrm{Re}\left[\frac{\Delta'}{(\omega'^2-\Delta'^2)^{1/2}}\right]\times\left\{\int d\omega_q\,\alpha^2(\omega_q)F(\omega_q)[D_q(\omega'+\omega)+D_q(\omega'-\omega)]-U_c\right\}, \tag{2}$$

where $D_q(\omega)=(\omega+\omega_q-i0^+)^{-1}$, $\Delta(\omega)=\varphi(\omega)/Z(\omega)$, and $\Delta_0=\Delta(\Delta_0)$. $F(\omega)$ is the phonon density of states

$$F(\omega)=\sum_\lambda\int\frac{d^3q}{(2\pi)^3}\delta(\omega-\omega_{q\lambda}), \tag{3}$$

and $\alpha^2(\omega)$ is an effective electron-phonon coupling function for phonons of energy ω,

$$\alpha^2(\omega)F(\omega)=\int_S d^2p\int_{S'}\frac{d^3p}{2\pi^2 v_F'}\sum_\lambda g_{pp'\lambda}\times\delta(\omega-\omega_{p-p'\lambda})\Big/\int_S d^2p, \tag{4}$$

where $g_{pp'\lambda'}$ is the dressed electron-phonon coupling constant, $\omega_{q\lambda}$ is the phonon energy for polarization λ and wave number q (reduced to the first zone), and v_F is the Fermi velocity. The two surface integrations are performed over the Fermi surface.

In Eq. (2) the ω' integration has been cut off at ω_c, and the Coulomb interaction has been replaced by a Coulomb pseudopotential given approximately[4,12,13] by

$$U_c=\frac{N(0)V_c}{[1+N(0)V_c\ln(E_F/\omega_c)]}, \tag{5}$$

where V_c is the static, screened Coulomb interaction averaged over the Fermi surface, and $N(0)$ is the electronic density of states at the Fermi surface unrenormalized by the electron-phonon interaction. Schrieffer, Scalapino, and Wilkins have shown that the (tunneling) electronic density of states in the superconductor is given by

$$\frac{N_s(\omega)}{N(0)}=\mathrm{Re}\left\{\frac{|\omega|}{[\omega^2-\Delta^2(\omega)]^{1/2}}\right\}, \tag{6}$$

and is measured directly in the superconductor–normal-metal tunnel junction at zero temperature. At finite temperature $T\ll T_c$, we have

$$\frac{(dI/dV)_s}{(dI/dV)_n}=\int\frac{N_s(\omega)}{N(0)}f'(V-\omega)d\omega, \tag{7}$$

where f' is the derivative of the Fermi function. One measures directly the electronic density

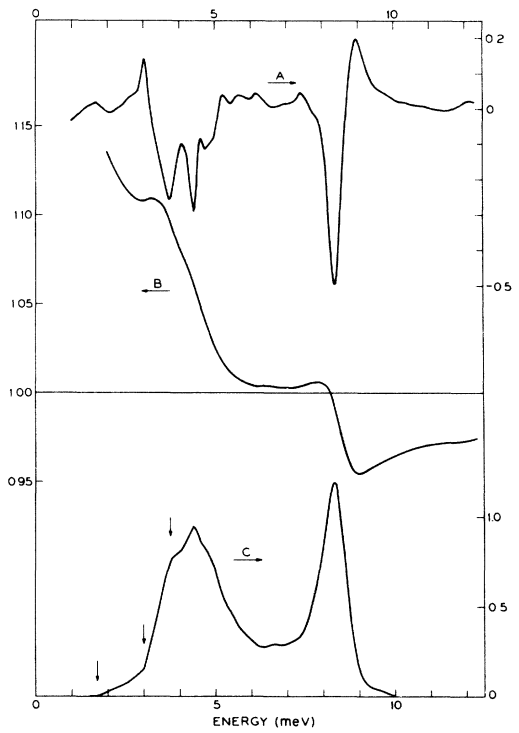

FIG. 1. Curve A is the normalized second derivative, $(d/dV)[(dI/dV)_S/(dI/dV)_n]$ (in units meV^{-1}) for a Pb-I-Pb junction at 0.8°K as a function of $V-2\Delta_0$. Curve B is the ratio of the (tunneling) electronic density of states of superconducting and normal lead as a function of $\omega-\Delta_0$. Curve C is $\alpha^2(\omega)F(\omega)$ (which is dimensionless) versus ω. The arrows indicate the singularities discussed in the text.

of states in the superconductor smeared by the Fermi distribution of the normal metal. We need to assume that T/T_C is small enough so that there are few quasiparticles excited in the superconductor and we can use the zero-temperature gap equation.

In order to eliminate the thermal smearing altogether we can use a two-superconductor tunnel junction where the current is proportional to the convolution of the density of states with itself,

$$I(V) \propto \int N_S(\omega) N_S(V-\omega) d\omega. \tag{8}$$

The convolution integral can be unfolded to yield $N_S(\omega)$ from $I(V)$. The electronic density of states determined in this way from the dV/dI-V characteristic of a Pb-I-Pb junction at ~0.8°K is shown as curve B in Fig. 1. These data agree with those from other Pb-I-Pb samples and with those obtained from a lead–normal-metal junction within ±0.002.[14] The second derivative of $I(V)$ for the same Pb-I-Pb junction is shown as curve A of Fig. 1. The two-superconductor data have no thermal smearing ($T < 0.12T_C$) and exhibit more detailed structure in the second derivative than superconductor–normal-metal data.

The information necessary to solve the gap equation and obtain the electronic density of states is the effective electron-phonon coupling function times the phonon density of states, $\alpha^2(\omega)F(\omega)$, and the Coulomb pseudopotential U_C. Schrieffer, Scalapino, and Wilkins chose the phonon density of states to be the sum of two Lorentzian peaks, a transverse one centered at 4.4 meV and a longitudinal one centered at 8.5 meV. The Coulomb pseudopotential was calculated from the free-electron model and Eq. (5) and was found to be 0.11 for $\omega_C = 32$ meV. The electron-phonon coupling function was assumed constant and was adjusted to yield the experimental value of Δ_0. They then solved the gap equation on a computer by iteration and compared the computed electronic density of states with the experimental one. We wish to reverse the direction of that calculation, start with the experimental electronic density of states $N_S(\omega)$ and energy gap Δ_0, and work backwards through the gap equation to obtain $\alpha^2(\omega)F(\omega)$ and U_C. The principle of this "inversion" of the gap equation is simple. We guess at $\alpha^2(\omega)F(\omega)$ and solve the gap equation, choosing U_C to give the experimental value of Δ_0, and calculate the electronic density of states $N_S^{\text{calc}}(\omega)$. We then use linear feedback techniques to correct $\alpha^2(\omega)F(\omega)$ for the "error" $N_S^{\text{expt}}(\omega) - N_S^{\text{calc}}(\omega)$. The correction to $\alpha^2(\omega)F(\omega)$ is

$$\delta[\alpha^2(\omega)] = \int d\omega' \left\{ \frac{\delta N(\omega')}{\delta[\alpha^2(\omega)F(\omega)]} \right\}^{-1} \times [N_S^{\text{expt}}(\omega') - N_S^{\text{calc}}(\omega')], \tag{9}$$

where $N_S^{\text{expt}}(\omega)$ is the experimental electronic density of states; the functional derivative can be calculated from the analytic form of the gap equation. Since the gap equation is nonlinear, this process must be iterated until it converges. $\alpha^2(\omega)F(\omega)$ is allowed to be an arbitrary function between 1 and 10 meV, and we find an $\alpha^2(\omega)F(\omega)$ and U_C which exactly fit Δ_0 and $N_S^{\text{expt}}(\omega)$ in the same energy range. The "harmonic" structure in $N_S^{\text{expt}}(\omega)$ at higher energies is accurately reproduced as well. The function $\alpha^2(\omega)F(\omega)$ which fits the electronic density-of-states curve shown in Fig. 1 is given as curve C of the Figure and is reproducible ±0.03. The transverse and longitudinal peaks in the phonon density of states expected from the work of Brockhouse et al.[15] are readily apparent in the $\alpha^2(\omega)F(\omega)$ curve. In addition, one can identify the well-developed shoulder in $\alpha^2(\omega)F(\omega)$ at 3.76 ± 0.05 meV with a Van Hove[16,17] singularity in $F(\omega)$ at 3.67 ± 0.08 meV due to transverse phonons near the zone boundary in the (111) direction.[15] The structure at 1.6 meV and 3.0 meV is too low in energy to be ascribed to Van Hove singularities in $F(\omega)$ and must be associated with structure in $\alpha^2(\omega)$. At low energies the longitudinal phonon density of states is 30 times smaller than the transverse density of states so that we need be concerned only with the transverse phonons. Since the Fermi surface of lead is nearly spherical, the long-wavelength transverse phonons are expected to be weakly coupled to the electrons. However, as soon as the wave number of the phonon becomes large enough to couple two sections of the Fermi surface via Umklapp processes, the transverse coupling function $\alpha_t^2(\omega)$ should show a discontinuity in slope and rise to a value comparable to the longitudinal coupling function. The structure at 1.6 and 3.0 meV is tentatively assigned to such a Fermi-surface effect on the effective coupling function for transverse phonons. One would expect a Kohn[15,18] anomaly

in the relevant transverse-phonon branch at the same energies. Van Hove singularities in the longitudinal peak and the end-point singularity at ~9 meV are not resolved. The Coulomb pseudopotential was found to be $U_C = 0.10 \pm 0.02$, in good agreement with the theoretical estimate of 0.11.[4,13] The electronic density of states in the normal metal which enters the expression for the heat capacity is renormalized by the electron-phonon interaction. This renormalization is given by Eq. (1) with Δ set equal to zero. One finds

$$Z_n(0) = 1 + 2\int_0^\infty \frac{\alpha^2(\omega)F(\omega)}{\omega} d\omega = 2.33 \pm 0.02.$$

The specific heat is $C = \gamma T = (2\pi^2/3)N(0)Z_n(0) \times k_B{}^2 T$; from the experimental value[19] $\gamma = 3.06$ mJ/mole-deg^2 and the above value of Z_n, we find that $N(0)$ is 88% of the free-electron value. The average of the effective electron-phonon coupling function is $\langle\alpha^2\rangle \equiv \int \alpha^2(\omega)F(\omega)d\omega / \int F(\omega)d\omega = (3.56 \pm 0.07 \text{ meV})/3N$.

We can separate the contributions of longitudinal and transverse phonons by estimating the longitudinal phonon density of states $F_l(\omega)$ from the Brockhouse data[15] and choosing a longitudinal coupling constant $\alpha_l{}^2$ so that $\alpha_l{}^2 F_l(\omega)$ approximately fits the peak in $\alpha^2(\omega)F(\omega)$ at ~8 meV. We find that the longitudinal phonons contribute 1.34 meV to $3N\langle\alpha^2\rangle$ so that the average longitudinal and transverse coupling constants are $N\alpha_l{}^2 \cong 1.34$ meV, $N\alpha_t{}^2 \cong 1.11$ meV (N = number of ions/cm^3). A theoretical estimate[22] of the coupling constants yields $N\alpha_l{}^2 = 1.98$ meV and $N\alpha_t{}^2 = 1.05$ meV.

The Pb-I-Pb tunnel junctions were prepared by heating freshly evaporated lead films in oxygen at ~50°C for 1½ h. The junction dimensions were 0.002 in. × 0.002 in. The normal metal of the M-I-Pb junctions was made by evaporating Al containing 5% Mn. This was oxidized in air for one minute, the junction dimensions being 0.008 in. × 0.008 in. Second-derivative measurements were made using the harmonic technique,[20] which gives $(d^2I/dV^2)\cdot(dI/dV)$; curve A of Fig. 1 is a true second derivative computed from such a measurement. The first derivative $(dV/dI)_S$ was measured in the manner described by Wyatt,[21] and $(dV/dI)_n$ was carefully taken with a magnetic field applied to the junction. The variation in $(dV/dI)_n$ with V accounts for differences between the density-of-states data of Fig. 1 and those reported earlier[10] where $(dV/dI)_n$ was assumed independent of voltage.

The gap Δ_0 for lead was taken equal to the bias where the first evidence of the aluminum gap appeared in the tunneling characteristics of an Al-I-Pb junction. We find $\Delta_0 = 1.358 \pm 0.004$ MeV.

The fact that one can fit the tunneling data with reasonable values of $\alpha^2(\omega)F(\omega)$ and U_C provides a confirmation of the theory of superconductivity with the retarded Eliashberg interaction and of the theory of electron tunneling. In addition, one now has a technique which will be applied to obtain information about the phonons and electron-phonon coupling constants for the more strongly coupled superconductors.

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VISCOSITY OF TYPE-II SUPERCONDUCTORS

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When the Lorentz driving force $(\vec{J}\times\vec{\varphi}_0)/c$ on the flux quanta in a type-II superconductor carrying a current $\vec{J}$ exceeds the pinning force, a voltage appears along the superconductor. This voltage has recently been measured by Kim et al.[1] and arises by induction from the flow of vortices driven across the superconductor by the transport current. The flow of vortices is opposed by viscous forces of the form $-\eta v_L$, where η is the viscosity coefficient and v_L the velocity of the vortex. The motion of vortices has been discussed by Anderson and Kim[2] and more recently by one of us,[3] who has shown that the only force acting on a vortex line is the Lorentz force $(\vec{J}\times\vec{\varphi}_0)/c$ arising out of the transport current. The object of this note is to discuss the viscosity coefficient η.

Kim and co-workers have shown that their experimental results over a wide range of temperature and composition can be described by the empirical formula

$$\eta_{\mathrm{emp}}=\pi\hbar H_{c2}\sigma/ec, \tag{1}$$

where H_{c2} is the upper critical field and σ the conductivity in the normal state. They suggest that the friction may result from currents flowing in the core of the vortex line, a cylinder with radius a approximately equal to the coherence distance ξ. This model follows from a calculation of Caroli, de Gennes, and Matricon,[4] who have shown that in this central region of the vortex the energy gap is so small that the conductivity is practically normal. It is found that η is much larger than can be accounted for by eddy currents resulting from the electric field $\vec{E}_m=-(1/c)\vec{v}_L\times\vec{H}$ generated by the moving magnetic field of the vortex line, which was used as the basis of a previous calculation by Volger, Staas, and Vijfeijken.[5] We show that there is another contribution to the electric field which can be much larger in the vicinity of the core and leads to an expression for η close to the empirical one.

We adopt a simple model for the vortex such that the core of radius a is normal with conductivity σ, and outside of the core the metal is superconducting. The materials used by Kim et al. are type-II superconductors with $\xi\ll\lambda$, where λ is the penetration depth. One may expect to describe these materials approximately by a local theory in which the current density is a function of $m\vec{v}_S(r_S)=\vec{p}_S(r)+(e/c)\times\vec{A}(r)$, where $\vec{p}_S$ is the common momentum of the paired electrons in the ground state and $\vec{A}(r)$ is the vector potential. The quantum condition for unit flux,

$$2\oint\vec{p}_S\cdot d\vec{l}=h,$$

gives $p_{S\theta}=\hbar/2r$. When the vortex line is moving, $\vec{r}$ is replaced by $\vec{r}-\vec{v}_Lt$; to a close approximation the current distribution is unmodified by the motion. However, an additional electric field beyond that from $\vec{A}(\vec{r}-\vec{v}_Lt)$ is required to change $\vec{v}_S$ with time as the vortex line moves past. This additional field is large near the core and is responsible for the major part of the energy loss when $\xi\ll\lambda$. The equation of motion for v_S, if we keep only the terms dependent on v_L, is[6]

$$m\,\partial\vec{v}_S/\partial t=-(\vec{v}_L\cdot\vec{\nabla})\vec{v}_S=-(e/m)\vec{E}. \tag{2}$$

The field may be expressed in the form[3]

$$\vec{E}=-(1/c)(\vec{v}_L\times\vec{H})-\vec{\nabla}\varphi, \tag{3}$$

where the electrostatic potential φ outside of the core is given by

$$-e\varphi=m\vec{v}_L\cdot\vec{v}_S. \tag{4}$$