NONLINEAR MODE INTERACTIONS IN UNIVERSAL PLASMA INSTABILITIES*

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Recent experimental investigations of the oscillations produced by the universal instability in a thermal plasma device (Q machine)[1-3] indicate that the plasma column oscillates as a bounded structure with traveling waves propagating in the azimuthal direction, corresponding to ascending values of the azimuthal mode number $l = 1, 2, 3, \cdots$,[1] and standing waves in the axial direction, corresponding to $\lambda_z \cong 2L$ (L is the column length).[2] A finite-length correction to the real part of the oscillation frequency (~10%) has also been observed.[3] These results, all in accord with theory,[4] suggest the possibility of regarding the plasma column as a bounded, spatially distributed oscillator with velocity-distributed constituent elements (plasma particles), reminiscent of the helium-neon laser.[5] The nonlinear behavior of the universal instability can then be analyzed conveniently by extending the methods of nonlinear mechanics[6] to distributed systems of this kind. This phenomenological approach is complementary to a more rigorous treatment and can also delineate the similarities and differences between the nonlinear behavior of the present plasma instability and that of other nonlinear oscillators. For example, the "mode-jumping" effect reported recently[3] is found to be related to mode-competition phenomena described in lumped-parameter vacuum-tube systems by van der Pol[7] and in lasers by Lamb,[8] and the analysis of the plasma case has benefitted greatly by the latter work.

We shall be interested here in nonlinear mode interactions in a two-mode system with incommensurate (i.e., not related by integer ratios) mode frequencies.

The time dependence of the mode energies is given by the "rate" equations

$$dE_1/dt = a_{10}E_1 - a_{11}E_1^2 - a_{12}E_1E_2,$$

$$dE_2/dt = a_{20}E_2 - a_{21}E_1E_2 - a_{22}E_2^2, \qquad (1)$$

where E_1 and E_2 are the two time-averaged mode energies and the a's are coefficients whose physical significance is explained below. (Unimportant numerical factors have been omitted.) These equations are obtained by writing general polynomial expansions of the form $dA_i/dt = P(A_i, A_j)$, where P denotes a cubic polynomial in A_i and A_j, the mode amplitudes, which are trigonometric functions of the time. Both sides of this equation are then multiplied by A_i, yielding $A_i(dA_i/dt) = \frac{1}{2}(d/dt)A_i^2 = \frac{1}{2}dE_i/dt$ on the left-hand side, and time averages are taken over intervals long compared with the mode periods. Because of the assumed orthogonality conditions, only the terms that appear in (1) survive when this procedure is carried out. These rate equations will be recognized as the generalization to a two-mode system of the expression suggested by Landau in connection with the analysis of turbulence.[9]

The a_{i0} represent the effective linear mode growth rates: $a_{i0} = \gamma_i^+ - \gamma_i^L$, where γ_i^L is the inherent growth rate while γ_i^L represents the ion Landau damping of the ith mode. The a_{ii} are nonlinear damping coefficients which would ordinarily represent any inherent plasma mechanism tending to saturate the instability. In the experiments described here, however, these terms have been introduced artificially by operating with electron sheaths at the endplates so that the instability is heavily damped by the Simon short-circuit effect.[1,10] The a_{ij} are mode-coupling coefficients and the negative signs are chosen in order to be consistent with the experimental results, as described below.

To investigate the stability of simultaneous oscillation in two modes we use phase-plane topology methods of nonlinear mechanics,[6] employing a phase plane whose coordinates are the two mode energies E_1 and E_2. The stability of an equilibrium point $E_1 = E_1^0$, $E_2 = E_2^0$ at which

$$dE_1/dt = dE_2/dt = 0, \qquad (2)$$

is determined by the nature of the singularities of the rate equations at this point. These singularities are examined by writing $E_i = E_i^0 + e_i$ ($e_i \ll E_i^0$), substituting in (1), neglecting quadratic terms in the e_i,[11] and using (2). This procedure yields

$$-\dot{e}_1 = (a_{11}E_1^0)e_1 + (a_{12}E_1^0)e_2,$$

$$-\dot{e}_2 = (a_{21}E_2^0)e_1 + (a_{22}E_2^0)e_2. \qquad (3)$$

The Routh-Hurwitz criterion[6] then provides the necessary conditions that must be satisfied for stable two-mode oscillation at E_1^0, E_2^0:

$$a_{11}a_{22} > a_{12}a_{21},$$
$$-(a_{11}+a_{22}) > 0; \qquad (4)$$

these conditions are achieved experimentally by adjustment of the sheath conditions, as explained above.

Mode competition can now be analyzed by determining the effect of varying an effective mode growth rate, say a_{10}, on the position of a stable equilibrium point (stable node), as shown in Fig. 1. The point denoted by I is a stable node, being the intersection of the two lines AA' and CC', which follow from (1) and (2):[8]

$$AA': \; E_2 = -(a_{11}/a_{12})E_1 + a_{10}/a_{12};$$
$$CC': \; E_2 = -(a_{21}/a_{22})E_1 + a_{20}/a_{22}. \qquad (5)$$

Assume that all parameters remain fixed, with the exception of a_{10}, which is now reduced to a'_{10}. When this is done AA' is translated into BB', and the stable node moves from I to II, as shown by the dashed arrow. We note that the steady-state amplitude of mode 1 is reduced while that of mode 2 is increased.

This displacement of a stable node can be demonstrated experimentally by using ion Landau damping to reduce a_{10} selectively, without affecting the other experimental parameters. The mechanism can be understood from the following. The phase velocity of a mode $v_\varphi = l(\omega^*/2\pi)2L$, where ω^* is a constant ($\omega^*/2\pi \sim 8$ kc/sec), while l and L have been defined above. When $v_\varphi < 3v_T^{(i)}$, where $v_T^{(i)}$ is the ion thermal velocity, the mode is damped by ion Landau damping since the negative curvature of the ion velocity distribution becomes important at these phase velocities. It has also been shown experimentally that the modes can be damped sequentially (in the order $l=1$, $l=2,\cdots$) when the column length is reduced gradually.[3] Thus, by appropriate adjustment of the sheath conditions and the column length, it is possible to set up an experimental situation in which only two modes oscillate.

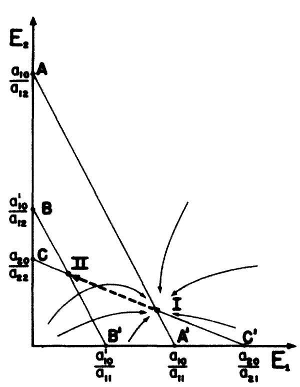

FIG. 1. Phase-plane diagram showing the displacement of a stable node from I to II when the effective growth rate of a mode is reduced. The coordinate axes are the mode energies in a two-mode system, and the existence of a stable node indicates that simultaneous oscillation in two modes is allowed. The intercepts are combinations of the experimental parameters: a_{i0} (effective growth rates), a_{ii} (damping coefficients), a_{ij} (mode-coupling coefficients).

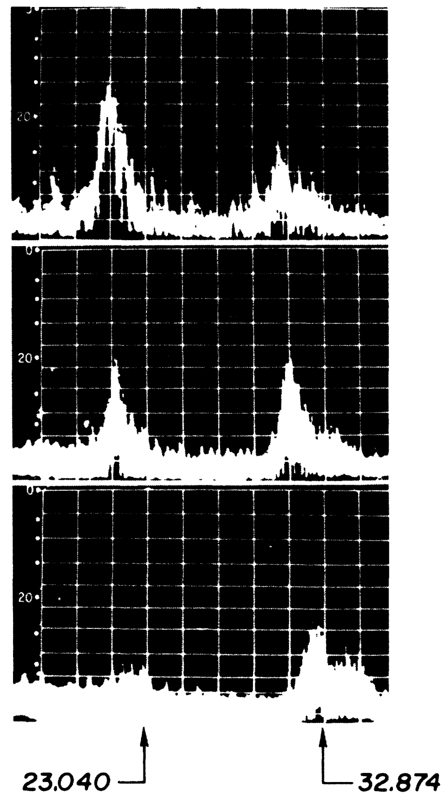

FIG. 2. Spectrum-analyzer presentation showing competition between azimuthal modes of the universal instability produced by gradually reducing the length of the plasma column to obtain selective mode quenching by ion Landau damping. The arrows denote frequency in kc/sec, and the modes correspond to $l=2$ and $l=3$ (l is the azimuthal mode number). The column length is 54, 46, and 38 cm in the top, center, and bottom photographs, respectively. The relevant phase velocities are as follows: top, $\nu_{\varphi 2} > 3\nu_T^{(i)}$, $\nu_{\varphi 3} > 3\nu_T^{(i)}$; center, $\nu_{\varphi 2} > 3\nu_T^{(i)}$, $\nu_{\varphi 3} > 3\nu_T^{(i)}$; bottom, $\nu_{\varphi 2} < 3\nu_T^{(i)}$ $\nu_{\varphi 3} \sim 3\nu_T^{(i)}$, where $\nu_T^{(i)}$ is the ion thermal velocity. Logarithmic vertical scale (full scale = 40 dB).

The results of such a two-mode experiment[12] are shown in Fig. 2, in which the $l=2$ and $l=3$ modes, respectively, play the roles of mode 1 and mode 2 in Fig. 1. In the top picture neither of these modes is Landau damped, but the $l=1$ mode (driven by the nonlinear interaction of the $l=2$ and $l=3$ modes) is heavily Landau damped; energy flows from the two high-frequency modes into the $l=1$ mode so that the mode-coupling coefficients a_{ij} in (1) are negative, as noted above.[13] In the center picture the $l=2$ mode is Landau damped, but the amplitude of the $l=3$ mode has increased, as predicted by Fig. 1. In the bottom picture the $l=2$ mode is quenched and the $l=3$ mode is starting to show the effects of Landau damping (displacement of the line CC' toward the origin in Fig. 1).

It is of interest to examine these results in the light of an observation by Lamb[8] that the possibility of multimode operation in the helium-neon laser depends on the fact that the constituent elements (excited atoms) are distributed in velocity, so that each mode can be driven by a different velocity class, with only a small amount of interaction (weak coupling). This is in contrast with single-stream or fluid-like strong-coupling systems (such as the van der Pol vacuum-tube oscillator) in which multimode oscillation is not allowed under the conditions described here (incommensurate frequencies and "soft" self-excitation[6]). The present plasma "microinstability" (with the artificial damping used in these experiments) is evidently similar to the laser case, with the different unstable modes being driven by different resonant-electron velocity classes; this feature is a specific consequence of the fact that the plasma electrons are distributed in velocity. It also appears that the collective effect of nonlinear mode interactions can have an important influence on individual mode amplitudes. Finally, this work points up the desirability of having available a theoretical calculation of the self-damping coefficients (a_{ii}) and the mode-coupling coefficients (a_{ii}) for the universal instability in the weakly nonlinear case.

A more detailed report of this work is in preparation.

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[13]Taking both a_{ij} to be negative is sufficient to recover the experimental results. In principle the a_{ij} can also be of opposite sign, but this case does not appear to apply here.

INFRARED ABSORPTION STRUCTURE IN RARE-EARTH METALS: RELATIONSHIP TO SPIN ARRANGEMENT AND BAND STRUCTURE

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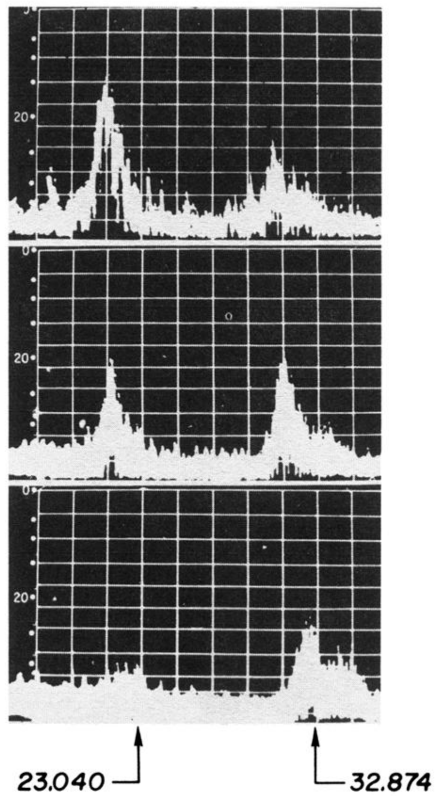
Recently Schüler[1] reported the appearance of structure at about 0.35 eV in the infrared reflectivity for a holmium film in the temperature range where holmium has a spiral spin arrangement with periodicity along the c axis. Following the ideas of Miwa,[2] Schüler suggested that the structure observed is due to the optical absorption corresponding to energy gaps in the conduction bands at the magnetic Brillouin-zone boundaries associated with the spiral periodicity. It is the purpose of the present note to point out (1) that there is a definitive experiment for distinguishing whether this structure, or indeed any part of the



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