and electromagnetic mass splittings are neglected throughout this calculation.

[4]M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1960).

[5]S. L. Adler, Phys. Rev. 137, B1022 (1965); and to be published.

[6]S. Fubini and G. Furlan, unpublished.

[7]The cancellation is exact at least at the double pole and in the region of small denominators where the unitarity condition can be continued off the mass shell.

[8]This use of the Low equation to manipulate the integration contours of the off-mass-shell scattering amplitude is similar to the approach of W. N. Cottingham, Ann. Phys. (N.Y.) 25, 424 (1963).

[9]G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1337 (1957). Our normalization and metric conventions differ from this reference.

[10]E. Ferrari and F. Selleri, Nuovo Cimento 21, 1028 (1961).

[11]For a more detailed treatment of possible off-mass-shell corrections, the reader is referred to S. L. Adler, accompanying Letter [Phys. Rev. Letters 14, 0000 (1965)].

[12]The relevant data has been tabulated by C. Hohlen, C. Ebel, and J. Giesecke, Z. Physik 180, 430 (1964). The author thanks Dr. M. Bander for his help in programming the numerical integrations.

[13]According to (24) it is the effect of the (3,3) resonance which makes $|G_A|>|G_V|$. In fact the (3,3) resonance contribution alone gives $|G_A/G_V|\simeq 1.3$, and the higher energy $T=\frac{1}{2}$ resonances reduce this value. The convergence of the integral depends on the validity of the Pomeranchuk theorem, but a $\sigma_{\mathrm{tot}}^{-}=C/\nu^{\alpha}$ fit to the data above 5 GeV with $\alpha=0.5$ to 0.7 gave a -0.02 contribution which has been included in the result.

CALCULATION OF THE AXIAL-VECTOR COUPLING CONSTANT RENORMALIZATION IN β DECAY

Stephen L. Adler*

Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts

(Received 17 May 1965)

1. Introduction.—We have derived a sum rule expressing the axial-vector coupling-constant renormalization in β decay in terms of off-mass-shell pion-proton total cross sections. This Letter briefly describes the derivation and gives the numerical results, which agree to within five percent with experiment. Full details will be published elsewhere.

The calculation is based on the following assumptions:

(A) The hadronic current responsible for $\Delta S=0$ leptonic decays is

$$J_\lambda = G_V(J_\lambda^{V1}+iJ_\lambda^{V2}+J_\lambda^{A1}+iJ_\lambda^{A2}), \tag{1}$$

where G_V is the Fermi coupling constant ($G_V\approx 1.02\times10^{-5}/M_N^2$).[1] Here $J_\lambda^{Va}=:\bar\psi_N\gamma_\lambda\frac{1}{2}\tau^a\times\psi_N+\cdots:$ is the vector current, which we assume to be the same as the isospin current,[2] and $J_\lambda^{Aa}=:\bar\psi_N\gamma_\lambda\gamma_5\frac{1}{2}\tau^a\psi_N+\cdots:$ is the axial-vector current. Since the vector current is conserved, the vector coupling constant is unrenormalized. The renormalized axial-vector coupling constant g_A is defined by

$$\langle N(q)|J_\lambda|N(q)\rangle = (M_N/q_0)G_V\bar u_N(q)(\gamma_\lambda+g_A\gamma_\lambda\gamma_5)\tau^+u_N(q). \tag{2}$$

(B) The axial-vector current is partially conserved (PCAC),[3]

$$\partial_\lambda J_\lambda^{Aa} = \frac{-iM_N M_\pi^2 g_A}{g_r K^{NN\pi}(0)}\varphi_\pi^{a}, \tag{3}$$

where g_r is the rationalized, renormalized pion-nucleon coupling constant ($g_r^2/4\pi\approx14.6$), $K^{NN\pi}(0)$ is the pionic form factor of the nucleon, normalized so that $K^{NN\pi}(-M_\pi^2)=1$, and φ_π^a is the renormalized pion field. According to Eq. (3), the chiralities $\chi^\pm(t)=\int d^3x(J_4^{A1}\pm iJ_4^{A2})$ satisfy

$$\frac{d}{dt}\chi^\pm(t) = \frac{\sqrt{2}M_N M_\pi^2 g_A}{g_r K^{NN\pi}(0)}\int d^3x\,\varphi_{\pi^\pm}. \tag{4}$$

(C) The axial-vector current satisfies the equal-time commutation relations

$$[J_4^{Aa}(x),J_4^{Ab}(y)]\Big|_{x_0=y_0} = \delta(\vec{x}-\vec{y})i\epsilon^{abc}J_4^{Vc}(x). \tag{5}$$

This implies that the chiralities satisfy

$$[\chi^+(t),\chi^-(t)] = 2I^3, \tag{6}$$

where I^3 is the third component of the isotopic spin.

The assumptions (A) are the usual ones for the leptonic decays. The additional hypotheses (B) and (C) are both necessary to obtain the sum rule for g_A. The hypotheses (A)-(C) are mutually consistent, in the sense that there is a renormalizable field theory (the σ model of Gell-Mann and Lévy[4]) in which they are exactly satisfied.

2. Sum rule. – There are two essentially equivalent ways to derive the sum rule for g_A. The first is to use a method proposed recently by Fubini and Furlan.[5] We take the matrix element of Eq. (6) between single proton states $\langle p(q)|$ and $|p(q')\rangle$. The right-hand side gives

$$\langle p(q)|2I^3|p(q')\rangle = (2\pi)^3\delta(\vec{q}-\vec{q}'). \tag{7}$$

In the matrix element of the commutator we insert a complete set of intermediate states, separating out the one-nucleon term (to which only the neutron contributes):

$$\langle p(q)|[\chi^+(t),\chi^-(t)]|p(q')\rangle$$
$$=\left\{\sum_{\text{spin}}\int\frac{d^3k}{(2\pi)^3}\langle p(q)|\chi^+(t)|n(k)\rangle\langle n(k)|\chi^-(t)|p(q')\rangle+\sum_{j\neq N}\langle p(q)|\chi^+(t)|j\rangle\langle j|\chi^-(t)|p(q')\rangle\right\}-(\chi^+\leftrightarrow\chi^-). \tag{8}$$

The one-neutron term is easily evaluated using Eq. (2), giving

$$(2\pi)^3\delta(\vec{q}-\vec{q}')g_A^{\ 2}(1-M_N^{\ 2}/q_0^{\ 2}). \tag{9}$$

In the summation over higher intermediate states we make use of Eq. (4), giving

$$\left[\frac{\sqrt{2}M_N M_\pi^{\ 2}g_A}{g_r K^{NN\pi}(0)}\right]^2\sum_{j\neq N}\frac{\langle p(q)|\int d^3x\,\varphi_{\pi^+}|j\rangle\langle j|\int d^3x\,\varphi_{\pi^-}|p(q')\rangle}{(q_0-q_{j0})^2}-(\pi^+\leftrightarrow\pi^-). \tag{10}$$

From Eqs. (9) and (10), we see that there is a family of sum rules, with q_0 as a parameter. In the limit as q_0 approaches infinity, a sum rule for $1-g_A^{\ -2}$ is obtained. Let us assume that the limiting operation can be taken inside the sum over intermediate states in Eq. (10). It is useful to write this sum in the form

$$\sum_{j\neq N}=\int\frac{d^3q_j}{(2\pi)^3}\int_{M_N+M_\pi}^\infty dW\sum_{\substack{j\neq N\\ \text{INT}}}\delta(W-M_j), \tag{11}$$

where q_j is the total momentum and where "INT" denotes the internal variables of the system j. The invariant mass of the system j is M_j. The integrations over x and q_j can be done explicitly, giving a factor $(2\pi)^3\delta(\vec{q}-\vec{q}')$, and constraining $\vec{q}_j$ to be equal to $\vec{q}$. Let us write

$$\langle j|\varphi_{\pi^\pm}(0)|p(q)\rangle=\left(\frac{M_N}{q_0}\frac{M_j}{q_{j0}}\right)^{1/2}F_j^{\ \pm}, \tag{12}$$

so that $F_j^\pm$ is a Lorentz scalar. Then using the facts that $q_{j0}=(q_0^{\ 2}+M_j^{\ 2}-M_N^{\ 2})^{1/2}$ and $(q_0-q_{j0})^{-2}=(q_0+q_{j0})^2/(M_j^{\ 2}-M_N^{\ 2})^2$, the limit of Eq. (10) becomes

$$\left[\frac{\sqrt{2}M_N g_A}{g_r K^{NN\pi}(0)}\right]^2(2\pi)^3\delta(\vec{q}-\vec{q}')\int_{M_N+M_\pi}^\infty\frac{dW\,M_N W}{(W^2-M_N^{\ 2})^2}$$
$$\times\lim_{q_0\to\infty}\left\{\frac{[q_0+(q_0^{\ 2}+W^2-M_N^{\ 2})^{1/2}]^2}{q_0(q_0^{\ 2}+W^2-M_N^{\ 2})^{1/2}}\right\}\lim_{q_0\to\infty}\{K^-[W,(q-q_j)^2]-K^+[W,(q-q_j)^2]\}, \tag{13a}$$

with

$$K^{\pm}[W,(q-q_j)^2]=\sum_{\substack{j\neq N\\ \text{INT}}}\delta(W-M_j)M_\pi^{\,4}|F_j^{\,\pm}|^2. \tag{13b}$$

The limit of the quantity in boldface curly brackets is 4, and the limit of the momentum transfer $(q-q_j)^2=-[q_0-(q_0^{\,2}+M_j^{\,2}-M_N^{\,2})^{1/2}]^2$ is 0. It is easy to see that $K^{\pm}(W,0)$ is equal to $[(W^2-M_N^{\,2})/(2\pi M_N)]\times\sigma_0^{\,\pm}(W)$, where $\sigma_0^{\,\pm}(W)$ is the total cross section for scattering of a zero-mass $\pi^{\pm}$ on a proton at center-of-mass energy W. Thus we get the simple and exact result

$$1-\frac{1}{g_A^{\,2}}=\frac{4M_N^{\,2}}{g_r^{\,2}K^{NN\pi}(0)^2}\frac{1}{\pi}\int_{M_N+M_\pi}^{\infty}\frac{WdW}{W^2-M_N^{\,2}}[\sigma_0^{\,+}(W)-\sigma_0^{\,-}(W)]. \tag{14}$$

Here $\sigma_0^{\,\pm}(W)$ is the total cross section for scattering of a zero-mass $\pi^{\pm}$ on a proton, at center-of-mass energy W.

The second method of getting the sum rule parallels the derivation from PCAC of a consistency condition on pion-nucleon scattering.[6] Using the identity

$$(d/dt)\langle N|T[\chi^a(t)\chi^b(0)]|N\rangle=\langle N|[\chi^a(t),\chi^b(0)]\delta(t)|N\rangle+\langle N|T[(d/dt)\chi^a(t)\chi^b(0)]|N\rangle, \tag{15}$$

and hypotheses (B) and (C), one obtains the relation

$$1-\frac{1}{g_A^{\,2}}=\frac{-2M_N^{\,2}}{g_r^{\,2}K^{NN\pi}(0)^2}G(0,0,0,0), \tag{16}$$

where

$$G(\nu,\nu_B,M_\pi^{\,i},M_\pi^{\,f})=\frac{1}{\nu}A^{\pi N(-)}(\nu,\nu_B,M_\pi^{\,i},M_\pi^{\,f})+B^{\pi N(-)}(\nu,\nu_B,M_\pi^{\,i},M_\pi^{\,f}). \tag{17}$$

Here A and B are the usual odd-isospin pion-nucleon scattering amplitudes, ν and ν_B are the energy and momentum transfer variables, and $M_\pi^{\,i}$ and $M_\pi^{\,f}$ are, respectively, the masses of the initial and final pion.[7] If $G(\nu,\cdots)$ is assumed to satisfy an unsubtracted dispersion relation in the energy variable ν, Eq. (14) follows from Eq. (17). Thus, the assumption that the limit $(q_0\to\infty)$ may be taken inside the sum over intermediate states in the method of Fubini and Furlan is equivalent to the assumption that $G(\nu,\cdots)$ obeys an unsubtracted dispersion relation. There is evidence that the unsubtracted dispersion relation for $G(\nu,\cdots)$ is valid.[8] Clearly, if a subtraction were required, the sum rule for g_A would be useless.

3. Numerical evaluation. —Because Eq. (14) involves off-mass-shell pion-proton scattering cross sections, a little work is necessary to compare it with experiment. Let us split the right-hand side of Eq. (14) into the sum of three terms:

$$1-\frac{1}{g_A^{\,2}}=\frac{4M_N^{\,2}}{g_r^{\,2}}(R_1+R_2+R_3), \tag{18}$$

with

$$R_1=-\frac{1}{\pi}\int_{M_\pi}^{\infty}\frac{d\nu}{d}\mathrm{Im}G(\nu,-M_\pi^{\,2}/2M_N,M_\pi,M_\pi)$$

$$=\frac{1}{2\pi}\int_{M_\pi}^{\infty}\frac{d\nu}{\nu^2}(\nu^2-M_\pi^{\,2})^{1/2}[\sigma^+(\nu)-\sigma^-(\nu)], \tag{19a}$$

$$R_2=\frac{1}{\pi}\int_{M_\pi}^{\infty}\frac{d\nu}{\nu}\mathrm{Im}G(\nu,-M_\pi^{\,2}/2M_N,M_\pi,M_\pi)-\frac{1}{\pi}\int_{M_\pi+M_\pi^{\,2}/2M_N}^{\infty}\frac{d\nu}{\nu}\mathrm{Im}G(\nu,0,M_\pi,M_\pi), \tag{19b}$$

and

$$R_3 = \frac{1}{\pi}\int_{M_\pi + M_\pi^2/2M_N}^{\infty} \frac{d\nu}{\nu} \mathrm{Im}\left[G(\nu,0,M_\pi,M_\pi) - \frac{G(\nu,0,0,0)}{K^{NN\pi}(0)^2}\right]. \quad (19c)$$

The dominant term, R_1, involves only the physical pion-proton total cross sections $\sigma^\pm$. Numerical evaluation gives[9,10]

$$(4M_N^2/g_r^2)R_1 = 0.254. \quad (20)$$

The term R_2 can be calculated in terms of pion-nucleon scattering phase shifts, giving[11]

$$(4M_N^2/g_r^2)R_2 = 0.155. \quad (21)$$

The term R_3, which describes corrections arising from taking the external pion off the mass shell, cannot be calculated directly from experimental data. In order to estimate this term, we assume that the off-mass-shell partial-wave amplitude $f_{lJI}(W, M_\pi^i, M_\pi^f)$ is given by

$$f_{lJI}(W, M_\pi^i, M_\pi^f) = \frac{f_{lJI}^{\mathrm{B}}(W, M_\pi^i, M_\pi^f)}{f_{lJI}^{\mathrm{B}}(W, M_\pi, M_\pi)} f_{lJI}(W, M_\pi, M_\pi). \quad (22)$$

(Here l = orbital angular momentum, J = total angular momentum, and I = isospin.) The superscript B denotes the Born approximation. Multiplying the physical f_{lJI} by the ratio of the Born approximations gives the off-mass-shell f_{lJI}, the correct threshold behavior, and the correct nearby left-hand singularities. Generalized unitarity implies that the off-mass-shell and the physical partial-wave amplitudes have nearly the same phase; Eq. (22), which gives them identical phases, approximately satisfies this requirement. Numerical evaluation of R_3, using Eq. (22), gives[12]

$$(4M_N^2/g_r^2)R_3 = -0.061. \quad (23)$$

It is possible that this number for R_3 is correct to within 20 %.[13]

Combining the three terms of Eq. (18) yields

$$g_A^{\text{theory}} = 1.24. \quad (24)$$

We have not attempted to make a detailed error estimate.[14] The best experimental value for g_A is[15]

$$g_A^{\text{expt}} = 1.18 \pm 0.02. \quad (25)$$

It is interesting that the region around the 600- and 900-MeV pion-nucleon resonances makes an important contribution to the sum rule. If only the contribution of the (3,3) resonance is retained, we get the result $g_A = 1.44$. Thus, the (3,3) resonance does not exhaust the sum rule.

After completing this work, I learned that a similar calculation has been done independently by Weisberger.[16]

---

*Junior Fellow, Society of Fellows.

[1]In the Cabibbo version of universality [N. Cabibbo, Phys. Rev. Letters 10, 531 (1963)], G_V is replaced by $\cos\theta G_V$.

[2]R. P. Feynman and M. Gell-Man, Phys. Rev. 109, 193 (1958).

[3]M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960); Y. Nambu, Phys. Rev. Letters 4, 380 (1960); S. L. Adler, Phys. Rev. 137, B1022 (1965).

[4]Gell-Mann and Lévy, reference 2. The viewpoint that the commutation relations of Eq. (5) may hold exactly is due to Gell-Mann [M. Gell-Mann, Physics 1, 63 (1964)].

[5]S. Fubini and G. Furlan, to be published.

[6]S. L. Adler, reference 3 and to be published; Y. Nambu and D. Lurié, Phys. Rev. 125, 1429 (1962); Y. Namby and E. Shrauner, Phys. Rev. 128, 862 (1962).

[7]In the scattering reaction $\pi(k_1)+p(q_1)\to\pi(k_2)+p(q_2)$, the variables ν, ν_B, M_π^i, and M_π^f are defined by $\nu = -k_1\cdot(q_1+q_2)/2M_N$, $\nu_B = k_1\cdot k_2/2M_N$, $(M_\pi^i)^2 = -k_1^2$, $(M_\pi^f) = -k_2^2$.

[8]First of all, the convergence of the sum rule of Eq. (11) suggests that an unsubtracted dispersion relation is valid. Secondly, B. Amblard et al., Phys. Letters 10, 138 (1964), have shown that the physical forward charge-exchange amplitude $G(\nu, -M_\pi^2/2M_N, M_\pi, M_\pi)$ satisfies an unsubtracted dispersion relation. It would be surprising if this result were changed by the extrapolation of the external pion mass from M_π to 0.

[9]Values of $\sigma^\pm$ from 0 to 110 MeV were taken from the smoothed fit of N. P. Klepikov et al., Joint Institute for Nuclear Research Report No. D-584, 1960 (unpublished). From 110 to 4950 MeV we used the tabulation of B. Amblard et al., Phys. Letters 10, 138 (1964) and private communication. Above 4950 MeV, we used the asymptotic formula $\sigma^- - \sigma^+ = 7.73\ \mathrm{mb} \times [k/$

$(1\ \text{BeV}/c)]^{-0.7}$ given by G. von Dardel et al., Phys. Rev. Letters 8, 173 (1962). This formula gives a good fit to the experimental data up to 20 BeV/c. The contribution to the sum rule of the region beyond 20 Bev/c is negligible.

[10]For the pion-nucleon coupling constant we used the value $f^2 = g_r^2 M_\pi^2/16\pi M_N^2 = 0.081 \pm 0.002$, quoted by W. S. Woolcock, Proceedings of the Aix-en-Provence Conference on Elementary Particles, 1961 (C.E.N., Saclay, France, 1961), Vol. I, p. 459.

[11]It is convenient to write R_2 as a single integral over pion-nucleon center-of-mass energy W, the integrand of which is the difference of two terms. This integral is sensitive only to low-energy pion-nucleon scattering data, since the two terms in the integrand cancel at high energies. The number quoted in the text was obtained using Roper's $l_m = 3$ phase shifts [L. D. Roper, Phys. Rev. Letters 12, 340 (1964), and private communication], truncating the integral at $W = 11.20\ M_\pi$. The integral is dominated by the (3, 3) resonances: Extending the integral only over the (3, 3) resonance gave $(4M_N^2/g_r^2)R_2 = 0.166$. A third calculation, using simple Breit-Wigner forms for the (3, 3) and the 600- and 900-MeV resonances, and neglecting all other partial waves, gave $(4M_N^2/g_r^2)R_2 = 0.156$. Thus, the value of R_2 is insensitive to "controversial" features of Roper's phases, such as whether the P_{11} wave resonates.

[12]This number was obtained using Roper's phase shifts, truncating the integral at $W = 11.20\ M_\pi$. Extending the integral only over the (3, 3) resonance gave $(4M_N^2/g_r^2)R_3 = -0.066$; evaluating the integral with only Breit-Wigner terms for the low-lying resonances gave $(4M_N^2/g_r^2)R_3 = -0.059$.

[13]To estimate the accuracy of the model, we repeated the calculation of R_3 with the assumption $f_{lJI}(W, 0, 0) = f_{lJI}(W, M_\pi, M_\pi)K^{NN\pi}(0)^2(W^2 - M_N^2)^{2l}[(W^2 - M_N^2 + M_\pi^2)^2 - 4W^2M_\pi^2]$, which includes only a threshold correction factor, and a constant factor $K^{NN\pi}(0)^2$ to account for the change in strength of the nearby left-hand singularities. The numerical result for $(4M_N^2/g_r^2)R_3$ was changed by about 20 %, to -0.051.

[14]The variation among different calculations (references 11-13) of R_2 and R_3 gives an idea of the uncertainty in the theoretical result.

[15]C. S. Wu, private communication.

[16]W. I. Weisberger, accompanying Letter [Phys. Rev. Letters 14, 0000 (1965)]. In the numerical evaluation of Weisberger, g_A is calculated from the dominant term R_l, giving $g_A = 1.16$.