RENORMALIZATION OF THE WEAK AXIAL-VECTOR COUPLING CONSTANT*

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It has been strongly suggested by Gell-Mann[1] that "the integrals of the time components of the vector and the axial-vector current octets ··· generate, under equal time commutation, the algebra of SU(3)⊗SU(3)," and that these algebraic relations are preserved even though the axial-vector and the strangeness-changing vector currents are not conserved. These nonlinear commutation relations fix the relative scale of the vector and axial-vector matrix elements measured in the weak interactions.

In this Letter these ideas are combined with that of a partially conserved $\Delta Y = 0$ axial-vector current to obtain an expression in terms of π-proton total cross sections, (24), for $|G_A/G_V|$, the absolute ratio of renormalized axial-vector and vector coupling constants of ordinary β decay. A numerical evaluation using experimental data for strong interaction π-nucleon scattering yields

$$|G_A/G_V| = 1.16. \tag{1}$$

The present experimental value[2] is

$$G_A/G_V = -1.18 \pm 0.02.$$

We consider the charges defined by

$$I^i = \int d^3x\, V_0^{\,i}, \quad Q_a^{\,i} = \int d^3x\, A_0^{\,i}, \quad i = 1, 2, 3, \tag{2}$$

where V_0^i, A_0^i are the time components of the isovector members of the vector and axial-vector current octets. $\vec{V}_\mu$ is, in fact, the conserved isotopic spin current[3] so that $\vec{I}$ is the total isotopic spin operator. $\vec{Q}_a$ is the isotopic chirality. The effective interaction for $\Delta Y = 0$ leptonic decays of the hadrons is taken as

$$-L_{\text{eff}} = (G_V/\sqrt{2}) j_{\text{lept}}^{\ \mu} (V_\mu^{\ +} \pm A_\mu^{\ +}) + \text{H.adj.},$$
$$V_\mu^{\ \pm} = V_\mu^{\ 1} \pm i V_\mu^{\ 2}. \tag{3}$$

For matrix elements between physical proton and neutron states of equal momentum it follows that

$$\langle P(p)|A_\mu^{\ +}(x)|N(p)\rangle = \frac{(M/E_p)}{(2\pi)^3}\frac{G_A}{G_V}\bar{u}(p)\gamma_\mu\gamma_5 u(p), \tag{4}$$

which defines G_A, the renormalized axial-vector coupling constant.

By partial conservation of the axial-vector current, (PCAC),[4,5] we mean

$$\partial^\mu A_\mu^{\ i}(x) = \frac{i\sqrt{2}M\mu^2}{g_{\pi n} K_{\pi nn}(0)} \varphi_\pi^{\ i}(x), \tag{5}$$

where $\varphi_\pi(x)$ is the renormalized Heisenberg field of the π mesons; M = nucleon mass, μ = pion mass, $g_{\pi n}^{\ 2}/4\pi = 14.6$, and $K_{\pi nn}(0)$ is the invariant π-nucleon vertex function evaluated at zero pion mass.

The commutation rule which we use is

$$2I_3 = [Q_a^{\ +}, Q_a^{\ -}]. \tag{6}$$

Adapting the method of Fubini and Furlan,[6] we take matrix elements of (6) between physical one-proton states which gives

$$\delta^{(3)}(\vec{p}_2 - \vec{p}_1) = \langle P(p_2)|[Q_a^{\ +}, Q_a^{\ -}]|P(p_1)\rangle. \tag{7}$$

We introduce a complete set of physical intermediate states in the right-hand side of (7) and isolate the contribution of one-neutron states:

$$\delta^{(3)}(\vec{p}_2 - \vec{p}_1) = \left(\frac{G_A}{G_V}\right)^2 \left[1 - \left(\frac{M}{E_p}\right)^2\right] \delta^{(3)}(\vec{p}_2 - \vec{p}_1) + \sum_{\alpha \neq N} \langle P(p_2)|Q_a^{\ +}|\alpha_{\text{out}}\rangle\langle_{\text{out}}\alpha|Q_a^{\ -}|P(p_1)\rangle$$
$$- \sum_\beta \langle P(p_2)|Q_a^{\ -}|\beta_{\text{out}}\rangle\langle_{\text{out}}\beta|Q_a^{\ +}|P(p_1)\rangle. \tag{8}$$

From (5)

$$\langle P(p_2)|Q_a^{\ +}|\alpha_{\text{out}}\rangle = \frac{-i\langle P(p_2)|\dot{Q}_a^{\ +}|\alpha_{\text{out}}\rangle}{E_p - E_\alpha} = \left[\frac{\sqrt{2}M\mu^2}{g_{\pi n}K_{\pi nn}(0)}\right] \int d^3x \frac{\langle P(p_2)|\varphi_\pi^{\ +}(x)|\alpha_{\text{out}}\rangle}{E_p - E_\alpha}. \tag{9}$$

We then obtain, for the second term on the right side of (8),

$$\sum_{\alpha\neq N}\langle P(p_2)|Q_a{}^+|\alpha_{\text{out}}\rangle\langle{}_{\text{out}}\alpha|Q_a{}^-|P(p_1)\rangle$$

$$=(2\pi)^6\delta^{(3)}(\vec{p}_2-\vec{p}_1)\left[\frac{\sqrt{2}M\mu^2}{g_{\pi n}K_{\pi nn}(0)}\right]^2\left(\frac{G_A}{G_V}\right)^2\int_{k_{0\,\min}}^{\infty}\frac{dk_0}{k_0{}^2}\sum_{\alpha\neq N}|\langle{}_{\text{out}}\alpha|\varphi_\pi{}^-(0)|P(p_1)\rangle|^2$$
$$\times\delta^{(3)}(\vec{p}_1-\vec{p}_\alpha)\delta(E_p+k_0-E_\alpha),\tag{10}$$

with

$$k_{0\,\min}=[(M+\mu)^2+|\vec{p}_1|^2]^{1/2}-E_p,\quad 0\leq k_{0\,\min}\leq\mu.$$

The multiplicative factor, $\delta^{(3)}(\vec{p}_2-\vec{p}_1)$, appears in all terms of (9) and will be dropped. We put $\vec{p}_2=\vec{p}_1=\vec{p}$. The matrix elements occuring in (10) can be classified into two types of Feynman diagrams: (a) connected graphs which correspond to scattering from an initial state of the proton and the off-mass-shell pion, in the rest frame of the pion, to the final state, $\langle{}_{\text{out}}\alpha|$; (b) disconnected graphs corresponding to propagation of the proton without interaction. For graphs of type (a),

$$\langle{}_{\text{out}}\alpha|\varphi_\pi{}^-(0)|P(p)\rangle_{\text{con.}}=-\frac{\langle{}_{\text{out}}\alpha|j_\pi{}^-(0)|P(p)\rangle}{k_0{}^2-\mu^2+i\epsilon},\tag{11}$$

where

$$j_\pi(x)\equiv(\Box+\mu^2)\varphi_\pi(x).$$

For graphs of type (b) we write

$$\langle{}_{\text{out}}\alpha|\varphi_\pi{}^-(0)|P(p)\rangle_{\text{disc.}}=\delta^{(3)}(\vec{p}-\vec{p}')\langle{}_{\text{out}}\alpha'|\varphi_\pi{}^-(0)|0\rangle,\tag{12}$$

where $\vec{p}'$ is the momentum of the free proton in the state $\langle{}_{\text{out}}\alpha|$.

When taking the absolute square of the matrix elements in (10) there will be contributions from squares of connected graphs, squares of disconnected graphs, and cross terms. All terms from squared disconnected graphs can be neglected since they will be exactly cancelled by corresponding contributions from the other term of the commutator in (8). The cross terms should be dominated by $|\alpha'\rangle=|\pi\rangle$, a physical one-pion state. Other states have higher thresholds in k_0, and their contribution should be damped strongly in the k_0 integration of (10). This assumption is in the basic spirit of the PCAC hypothesis. One then obtains

$$\int_{k_{0\,\min}}^{\infty}\frac{dk_0}{k_0{}^2}\sum_{\alpha\neq N}|\langle{}_{\text{out}}\alpha|\varphi_\pi{}^-(0)|P(p)\rangle|^2\delta^{(3)}(\vec{p}-\vec{p}_\alpha)\delta(E_p+k_0-E_\alpha)$$

$$=\int_{k_{0\,\min}}^{\infty}\frac{dk_0}{k_0{}^2}\left\{\frac{|\langle{}_{\text{out}}\alpha|j_\pi{}^-(0)|P\rangle|^2\delta^{(3)}(\vec{p}-\vec{p}_\alpha)\delta(E_p+k_0-E_\alpha)}{(k_0{}^2-\mu^2+i\epsilon)(k_0{}^2-\mu^2-i\epsilon)}-\frac{\delta(k_0-\mu)}{(2\pi)^{3/2}\sqrt{2}\mu}\left[\frac{{}_{\text{out}}P(p)\pi^-(\vec{k}=0)|j_\pi{}^-(0)|P(p)\rangle}{k_0{}^2-\mu^2+i\epsilon}\right.\right.$$
$$\left.\left.+\frac{\langle P|j_\pi{}^+(0)|P(p)\pi^-(\vec{k}=0)_{\text{out}}\rangle}{k_0{}^2-\mu^2-i\epsilon}\right]\right\}.\tag{13}$$

The frightening looking singularities in the above expression will be cancelled by unitarity. Since the T-matrix elements in (13) are multiplied by $\delta(k_0-\mu)$, they can be continued in the pion mass from μ to k_0. That is,

$$\frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}\mu}\langle{}_{\text{out}}P(p)\pi^-(\vec{k}=0)|j_\pi{}^-(0)|P(p)\rangle\xrightarrow[\mu\to k_0]{}\frac{1}{(2\pi)^3(2k_0)}T_{\pi^-p}(k_0,E_p);$$

$$T_{\pi^- p}(k_0, E_p) = \int d^4y\, e^{ik_0 y_0} \langle P(p) | T[j_\pi^+(y) j_\pi^-(0)] | P(p) \rangle$$

$$+ \text{(equal-time commutators independent of } k_0\text{)}. \tag{14a}$$

$T[j_\pi^+(y)j_\pi^-(0)]$ denotes the time-ordered product of the pion currents. Similarly,

$$\frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\mu}} \langle P | j_\pi^+(0) | P\pi^-(0)_{\text{out}} \rangle \xrightarrow[\mu \to k_0]{} \frac{1}{(2\pi)^3 (2k_0)} T_{\pi^- p}^\dagger(k_0, E_p). \tag{14b}$$

$T_{\pi^- p}^\dagger$ has a representation similar to that for $T_{\pi^- p}$ above, except in terms of an antitime-ordered product. $T_{\pi^- p}(k_0, E_p)$ is the π^--proton forward elastic scattering amplitude in reference system where the off-mass-shell pion is at rest and the proton has momentum p. If we now use

$$\delta(k_0 - \mu) = \frac{k_0}{\pi i}\left[\frac{1}{k_0^2 - \mu^2 - i\epsilon} - \frac{1}{k_0 - \mu^2 + i\epsilon}\right],$$

and substitute these results into (13), it can be seen that the coefficient of the dangerous double-pole pinch vanishes due to the unitarity condition.[7]

Performing the same manipulations for the $\sum_\beta$ term in (9), one arrives at the result

$$\left(\frac{G_A}{G_V}\right)^{-2} = 1 - \left(\frac{M}{E_p}\right)^2 + (2\pi)^3 \left[\frac{2M\mu^2}{g_{\pi n} K_{\pi nn}(0)}\right]^2 \left(\frac{1}{2\pi i}\right) \int_{k_{0\,\min}}^{\infty} \frac{dk_0}{k_0^2} \left[\frac{T_{\pi^- p}(k_0, E_p)}{(k_0^2 - \mu^2 + i\epsilon)^2} - \frac{T_{\pi^- p}^\dagger(k_0, E_p)}{(k_0^2 - \mu^2 - i\epsilon)^2} - \frac{T_{\pi^+ p}(k_0, E_p)}{(k_0^2 - \mu^2 + i\epsilon)^2} + \frac{T_{\pi^+ p}^\dagger(k_0, E_p)}{(k_0^2 - \mu^2 - i\epsilon)^2}\right] \tag{15}$$

To evaluate the integral above, we return to the expression (14a) for $T_{\pi^- p}(k_0, E_p)$, insert a complete set of intermediate states in the time-ordered product, and obtain a Low equation for T:

$$T_{\pi^- p}(k_0, E) = (2\pi)^3 \left\{ \sum_\alpha \frac{|\langle P(p) | j_\pi^+ | \alpha \rangle|^2 \delta^{(3)}(\vec{p} - \vec{p}_\alpha)}{k_0 + E_p - E_\alpha + i\epsilon} - \sum_\beta \frac{|\langle P(p) | j_\pi^- | \beta \rangle|^2 \delta^{(3)}(\vec{p} - \vec{p}_\alpha)}{k_0 - E_p + E_\alpha - i\epsilon} \right\}. \tag{16}$$

All the k_0 dependence of $T_{\pi^- p}$ is in the denominators. For fixed physical E_p, $T_{\pi^- p}(k_0, E_p)$ is analytic in the complex k_0 plane with branch cuts from $k_{0\,\min}$ to $+\infty$ and from $-k_{0\,\min}$ to $-\infty$. The one-nucleon intermediate state does not contribute a pole term at $k_0 = 0$ since the residue is zero for a pseudoscalar pion. Let $\mathcal{T}(z, E_p)$ denote the analytic continuation of $T_{\pi^- p}(k_0, E_p)$. $T_{\pi^- p}(k_0, E_p)$ is the limit of $\mathcal{T}(z, E_p)$ on the top of the right-hand cut and the bottom of the left-hand cut. It can be shown in the same manner that $T_{\pi^- p}^\dagger(k_0, E_p)$ is the limit of $\mathcal{T}(z, E_p)$ on the other sides of the cuts. Using crossing symmetry, the integral[8] in (15) can then be evaluated as

$$\frac{1}{2\pi i} \oint_C \frac{\mathcal{T}(z, E_p)}{(z^2 - \mu^2)^2} \frac{dz}{z^2} = \frac{1}{\mu^4} \frac{d}{dk_0} T_{\pi^- p}(k_0, E_p) \Big|_{k_0 = 0}, \tag{17}$$

where C is the contour indicated in Fig. 1.

Crossing symmetry further implies that

$$\frac{d}{dk_0} T_{\pi^- p}(k_0, E_p) \Big|_{k_0 = 0} = \frac{d}{dk_0} T^-(k_0, E_p) \Big|_{k_0 = 0}, \tag{18}$$

where T^- is the coefficient of the antisymmetric isospin function in the conventional decomposition for π-nucleon scattering.[9]

We seek to reduce our answer to an expression involving on-the-mass-shell quantities only. The forward scattering amplitudes satisfy dispersion relations[9] in the variables ν, the pion energy in the "laboratory system" where the nucleon is at rest, and these dispersion relations can be continued in the pion mass to $k^2=0$. Since $k_0E_p=M\nu$,

$$\frac{d}{dk_0}T^-(k_0,E_p)\Big|_{k_0=0}=\left(\frac{\partial}{\partial k_0}+\frac{E}{M}\frac{\partial}{\partial\nu}\right)T^-(k_0,\nu)\Big|_{k_0=\nu=0},\quad \nu/k=E/M. \tag{19}$$

From the dispersion relations,

$$(2\pi)^3\frac{d}{dk_0}T^-(k_0,E_p)\Big|_{k_0=0}=\left[\frac{g_{\pi n}K_{\pi nn}(0)}{\sqrt{2}E_p}\right]^2+\frac{2}{\pi}\int_{\mu+\mu^2/2M}^{\infty}\frac{d\nu}{\nu^2}\mathrm{Im}\mathfrak{M}^-(k_0=0,\nu), \tag{20}$$

where $\mathfrak{M}$ is the invariant forward scattering amplitude. The energy-dependent term contribution comes from the Born term. We now write for (15)

$$\left(\frac{G_A}{G_V}\right)^{-2}=1+(1/\pi)[(2M)/g_{\pi n}K_{\pi nn}(0)]^2\int_{\mu+\mu^2/2M}^{\infty}\frac{d\nu}{\nu^2}\mathrm{Im}\mathfrak{M}^-(k_0=0,\nu). \tag{21}$$

The contribution of the Born term from the dispersion relations has completely cancelled the original factor $(M/E_p)^2$ from the one-neutron intermediate state, and we are left with a covariant answer. To put the answer in a final useful form, first let $\nu_L=\nu-(\mu^2/2M)$. In the region of integration,[10,11]

$$\mathfrak{M}^-[k_0=0,\nu_L+(\mu^2-k_0^{\,2})/2M]\simeq K_{\pi nn}^{\;2}(0)\mathfrak{M}^-(\mu,\nu_L), \tag{22}$$

$$\mathrm{Im}\mathfrak{M}^-(\mu,\nu_L)=q\sigma_{\mathrm{tot}}^{\;-}(\nu_L), \tag{23}$$

where q is the magnitude of the three-momentum of the pion in the laboratory system. We obtain finally an answer in terms of experimentally measured total cross sections,

$$\left|\frac{G_A}{G_V}\right|=\left\{1+\left(\frac{2}{\pi}\right)\left(\frac{M}{g_{\pi n}}\right)\int_{\mu}^{\infty}\frac{q\,d\nu_L}{\nu_L^{\,2}}[\sigma_{\mathrm{tot}}^{\pi^-p}(\nu_L)-\sigma_{\mathrm{tot}}^{\pi^+p}(\nu_L)]\right\}^{-1/2}. \tag{24}$$

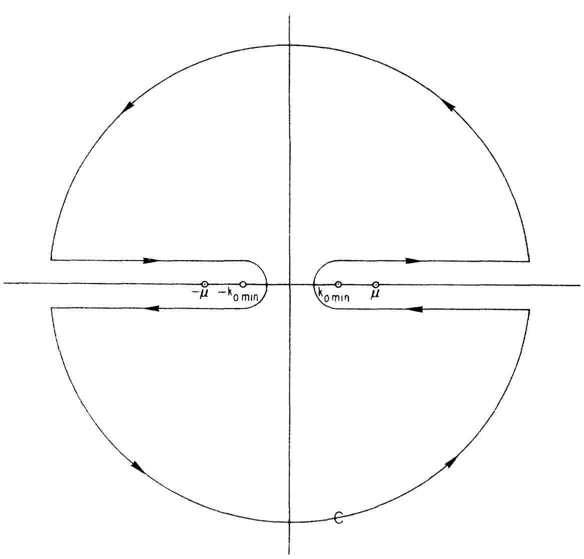

FIG. 1. The contour of integration C for the integral in (17).

Evaluation of (24) using experimental cross sections[12] gives the result quoted in (1).[13]

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After completing this work, the author was informed that similar results have been independently obtained by Adler.[11]

[1]M. Gell-Mann, Phys. Rev. 125, 1067 (1962); Physics 1, 63 (1964).

[2]C. S. Wu, private communication.

[3]Electromagnetic violations of isospin conservation

and electromagnetic mass splittings are neglected throughout this calculation.

[4]M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1960).

[5]S. L. Adler, Phys. Rev. 137, B1022 (1965); and to be published.

[6]S. Fubini and G. Furlan, unpublished.

[7]The cancellation is exact at least at the double pole and in the region of small denominators where the unitarity condition can be continued off the mass shell.

[8]This use of the Low equation to manipulate the integration contours of the off-mass-shell scattering amplitude is similar to the approach of W. N. Cottingham, Ann. Phys. (N.Y.) 25, 424 (1963).

[9]G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1337 (1957). Our normalization and metric conventions differ from this reference.

[10]E. Ferrari and F. Selleri, Nuovo Cimento 21, 1028 (1961).

[11]For a more detailed treatment of possible off-mass-shell corrections, the reader is referred to S. L. Adler, accompanying Letter [Phys. Rev. Letters 14, 0000 (1965)].

[12]The relevant data has been tabulated by C. Hohlen, C. Ebel, and J. Giesecke, Z. Physik 180, 430 (1964). The author thanks Dr. M. Bander for his help in programming the numerical integrations.

[13]According to (24) it is the effect of the (3,3) resonance which makes $|G_A|>|G_V|$. In fact the (3,3) resonance contribution alone gives $|G_A/G_V|\simeq 1.3$, and the higher energy $T=\frac{1}{2}$ resonances reduce this value. The convergence of the integral depends on the validity of the Pomeranchuk theorem, but a $\sigma_{\rm tot}^{-}=C/\nu^{\alpha}$ fit to the data above 5 GeV with $\alpha=0.5$ to 0.7 gave a -0.02 contribution which has been included in the result.

CALCULATION OF THE AXIAL-VECTOR COUPLING CONSTANT RENORMALIZATION IN β DECAY

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1. Introduction.—We have derived a sum rule expressing the axial-vector coupling-constant renormalization in β decay in terms of off-mass-shell pion-proton total cross sections. This Letter briefly describes the derivation and gives the numerical results, which agree to within five percent with experiment. Full details will be published elsewhere.

The calculation is based on the following assumptions:

(A) The hadronic current responsible for $\Delta S=0$ leptonic decays is

$$J_\lambda=G_V(J_\lambda^{V1}+iJ_\lambda^{V2}+J_\lambda^{A1}+iJ_\lambda^{A2}), \qquad (1)$$

where G_V is the Fermi coupling constant ($G_V\approx 1.02\times 10^{-5}/M_N^2$).[1] Here $J_\lambda^{Va}=:\bar\psi_N\gamma_\lambda\frac{1}{2}\tau^a\times\psi_N+\cdots:$ is the vector current, which we assume to be the same as the isospin current,[2] and $J_\lambda^{Aa}=:\bar\psi_N\gamma_\lambda\gamma_5\frac{1}{2}\tau^a\psi_N+\cdots:$ is the axial-vector current. Since the vector current is conserved, the vector coupling constant is unrenormalized. The renormalized axial-vector coupling constant g_A is defined by

$$\langle N(q)|J_\lambda|N(q)\rangle = (M_N/q_0)G_V\bar u_N(q)(\gamma_\lambda+g_A\gamma_\lambda\gamma_5)\tau^+u_N(q). \qquad (2)$$

(B) The axial-vector current is partially conserved (PCAC),[3]

$$\partial_\lambda J_\lambda^{Aa}=\frac{-iM_NM_\pi^2g_A}{g_rK^{NN\pi}(0)}\varphi_\pi^a, \qquad (3)$$

where g_r is the rationalized, renormalized pion-nucleon coupling constant ($g_r^2/4\pi\approx 14.6$), $K^{NN\pi}(0)$ is the pionic form factor of the nucleon, normalized so that $K^{NN\pi}(-M_\pi^2)=1$, and φ_π^a is the renormalized pion field. According to Eq. (3), the chiralities $\chi^\pm(t)=\int d^3x(J_4^{A1}\pm iJ_4^{A2})$ satisfy

$$\frac{d}{dt}\chi^\pm(t)=\frac{\sqrt2 M_NM_\pi^2g_A}{g_rK^{NN\pi}(0)}\int d^3x\,\varphi_{\pi^\pm}. \qquad (4)$$

(C) The axial-vector current satisfies the equal-time commutation relations

$$[J_4^{Aa}(x),J_4^{Ab}(y)]\Big|_{x_0=y_0}=\delta(\vec x-\vec y)i\epsilon^{abc}J_4^{Vc}(x). \qquad (5)$$

This implies that the chiralities satisfy

$$[\chi^+(t),\chi^-(t)]=2I^3, \qquad (6)$$

where I^3 is the third component of the isotopic spin.