Editors' Suggestion

Cluster State as a Noninvertible Symmetry-Protected Topological Phase

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We show that the standard $1+1$D $\mathbb{Z}_2\times\mathbb{Z}_2$ cluster model has a noninvertible global symmetry, described by the fusion category $\mathrm{Rep}(\mathrm{D}_8)$. Therefore, the cluster state is not only a $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry protected topological (SPT) phase, but also a noninvertible SPT phase. We further find two new commuting Pauli Hamiltonians for the other two $\mathrm{Rep}(\mathrm{D}_8)$ SPT phases on a tensor product Hilbert space of qubits, matching the classification in field theory and mathematics. We identify the edge modes and the local projective algebras at the interfaces between these noninvertible SPT phases. Finally, we show that there does not exist a symmetric entangler that maps between these distinct SPT states.

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Introduction—Symmetry protected topological (SPT) phases [1–6] are some of the most fundamental quantum phases of matter. Without imposing any global symmetry, these phases are gapped with a unique, nondegenerate ground state, and are completely featureless. However, they become distinct topological states when we impose a symmetry G, in the sense that there cannot be a continuous G-symmetric deformation connecting these states without a phase transition. See Refs. [7–9] for reviews.

The simplest example of an SPT phase is the $1+1$d cluster Hamiltonian [10,11]:

$$H_{\text{cluster}} = -\sum_{j=1}^{L} Z_{j-1}X_jZ_{j+1}. \tag{1}$$

We assume the space is a closed periodic chain of L qubits with even L. It has a $\mathbb{Z}_2^{\mathrm{e}}\times\mathbb{Z}_2^{\mathrm{o}}$ symmetry generated by

$$\eta^{\mathrm{e}} = \prod_{j:\text{even}} X_j, \qquad \eta^{\mathrm{o}} = \prod_{j:\text{odd}} X_j. \tag{2}$$

The Hamiltonian is gapped with a unique ground state, known as the cluster state, satisfying $Z_{j-1}X_jZ_{j+1}|\text{cluster}\rangle = |\text{cluster}\rangle$. Indeed, it is a commuting projector of Pauli operators with no relation, and therefore the ground state degeneracy is 1.

The cluster state is in a distinct $\mathbb{Z}_2^{\mathrm{e}}\times\mathbb{Z}_2^{\mathrm{o}}$ SPT phase compared to the product state $|++\cdots+\rangle$ (which is the ground state of the trivial Hamiltonian $H_{\text{trivial}} = -\sum_{j=1}^{L} X_j$). Explicitly, it is given by

$$|\text{cluster}\rangle = \mathsf{V}|++\cdots+\rangle, \quad \text{where } \mathsf{V} = \prod_{j=1}^{L} \mathrm{CZ}_{j,j+1}. \tag{3}$$

Here $\mathrm{CZ}_{j,j+1} = [(1+Z_j+Z_{j+1}-Z_jZ_{j+1})/2]$. Since V, known as the cluster entangler, is a finite-depth circuit, the cluster state is in the same phase as the product state if we do not impose any global symmetry. However, if we impose the $\mathbb{Z}_2^{\mathrm{e}}\times\mathbb{Z}_2^{\mathrm{o}}$ symmetry, the cluster state is a distinct SPT state compared to the (trivial) product state [9]. Indeed, while V globally is $\mathbb{Z}_2^{\mathrm{e}}\times\mathbb{Z}_2^{\mathrm{o}}$ symmetric, the individual gates $\mathrm{CZ}_{j,j+1}$ are not.

In recent years, the notion of global symmetry has been generalized in many different directions [12]. In particular, there has been a lot of progress on a novel kind of symmetry, known as noninvertible symmetries, in quantum field theory and condensed matter theory. Noninvertible symmetries are implemented by conserved operators without inverses, and therefore are not described by group theory. See Refs. [13–17] for recent reviews.

The critical Ising lattice model serves as the prototypical example of a gapless system with a noninvertible symmetry, associated with the Kramers-Wannier duality [18–24]. However, this noninvertible symmetry has a Lieb-Schultz-Mattis-type constraint, implying that it is incompatible with a unique gapped ground state [24,25]. It is then natural to ask what is the simplest nondegenerate gapped phase protected by a noninvertible symmetry.

In this Letter we show that the standard cluster Hamiltonian (1) has a noninvertible symmetry. Therefore, the cluster state is not only a $\mathbb{Z}_2^{\mathrm{e}}\times\mathbb{Z}_2^{\mathrm{o}}$ SPT phase, but also a *noninvertible SPT phase*, i.e., a nondegenerate gapped phase invariant under a noninvertible symmetry. We furthermore find two other distinct SPT states protected by the same noninvertible symmetry, matching the expectation from category theory [26] and field theory [27].

Noninvertible symmetry of the cluster model—The key observation is that the cluster Hamiltonian is invariant under the transformation (Throughout the Letter, we use $\rightsquigarrow$ to denote a transformation implemented by a noninvertible operator, and $\mapsto$ for a conventional symmetry transformation implemented by a unitary operator.)

$$X_j \rightsquigarrow Z_{j-1}Z_{j+1}, \qquad Z_{j-1}Z_{j+1} \rightsquigarrow X_j. \tag{4}$$

However, suppose this transformation were implemented by an invertible operator U such that $UX_jU^{-1} = Z_{j-1}Z_{j+1}$; then $U\eta^{\mathrm{e}}U^{-1} = \prod_{j:\mathrm{even}} Z_{j-1}Z_{j+1} = 1$, which is a contradiction. Instead, this transformation is implemented by the following conserved operator:

$$\mathsf{D} = T\mathsf{D}^{\mathrm{e}}\mathsf{D}^{\mathrm{o}}, \tag{5}$$

where

$$\begin{aligned}\mathsf{D}^{\mathrm{e}} &= e^{\frac{2\pi iL}{16}}\frac{1+\eta^{\mathrm{e}}}{\sqrt{2}}\frac{1-iX_L}{\sqrt{2}}\cdots\frac{1-iZ_4Z_2}{\sqrt{2}}\frac{1-iX_2}{\sqrt{2}},\\ \mathsf{D}^{\mathrm{o}} &= e^{\frac{2\pi iL}{16}}\frac{1+\eta^{\mathrm{o}}}{\sqrt{2}}\frac{1-iX_{L-1}}{\sqrt{2}}\cdots\frac{1-iZ_3Z_1}{\sqrt{2}}\frac{1-iX_1}{\sqrt{2}},\end{aligned} \tag{6}$$

are the Kramers-Wannier operators on the even and odd sites in [23,24]. Here T is the lattice translation by one site which acts on local operators as $TX_jT^{-1} = X_{j+1}$, $TZ_jT^{-1} = Z_{j+1}$ and satisfies $T^L = 1$. We have $T\mathsf{D}^{\mathrm{e}} = \mathsf{D}^{\mathrm{o}}T$ and $T\mathsf{D} = \mathsf{D}T$. However, D is not a conventional symmetry operator because it has a kernel—it annihilates every state that is not $\mathbb{Z}_2^{\mathrm{e}} \times \mathbb{Z}_2^{\mathrm{o}}$ symmetric. It implements (4) in the sense that

$$\mathsf{D}X_j = Z_{j-1}Z_{j+1}\mathsf{D}, \qquad \mathsf{D}Z_{j-1}Z_{j+1} = X_j\mathsf{D}. \tag{7}$$

The operators η^{e}, η^{o}, and D commute with the Hamiltonian and satisfy the following algebra:

$$\begin{aligned}&\mathsf{D}^2 = 1 + \eta^{\mathrm{e}} + \eta^{\mathrm{o}} + \eta^{\mathrm{e}}\eta^{\mathrm{o}},\\ &\eta^{\mathrm{e}}\mathsf{D} = \mathsf{D}\eta^{\mathrm{e}} = \eta^{\mathrm{o}}\mathsf{D} = \mathsf{D}\eta^{\mathrm{o}} = \mathsf{D}.\end{aligned} \tag{8}$$

Note that even though the definition of D appears to involve a lattice translation T, the factors D^{e}, D^{o} each involve a half translation in the opposite direction, and hence the algebra of D does not mix with lattice translations. We conclude that the cluster state is a topological phase protected by a noninvertible symmetry.

In the Supplemental Material [28], we show that this noninvertible symmetry, together with the η's, are described by the fusion category $\mathrm{Rep}(\mathrm{D}_8)$, whose fusion algebra is given by the tensor product of the irreducible representations of the group D_8. We prove this by gauging the noninvertible symmetry and find a dual D_8 symmetry.

Noninvertible SPT phases—We have shown that the cluster state is a noninvertible SPT phase. Are there other $\mathrm{Rep}(\mathrm{D}_8)$ SPT phases? Note that D does not act on site, and the product state is not invariant under it. Mathematically, an SPT phase (invertible or not) corresponds to a fiber functor of the fusion category. It is known [26] that there are three distinct fiber functors for $\mathrm{Rep}(\mathrm{D}_8)$, labeled by the three nontrivial elements $\eta^{\mathrm{e}}, \eta^{\mathrm{o}}, \eta^{\mathrm{d}} = \eta^{\mathrm{e}}\eta^{\mathrm{o}}$ of $\mathbb{Z}_2^{\mathrm{e}} \times \mathbb{Z}_2^{\mathrm{o}}$. See Refs. [27,29–32] for the field theory discussions.

What are the lattice models for the other two $\mathrm{Rep}(\mathrm{D}_8)$ SPT phases? To find them, we partially gauge the noninvertible symmetry and look for different spontaneous symmetry breaking patterns in the gauged theory. We first review this method in the case of $\mathbb{Z}_2^{\mathrm{e}} \times \mathbb{Z}_2^{\mathrm{o}}$.

For $\mathbb{Z}_2^{\mathrm{e}} \times \mathbb{Z}_2^{\mathrm{o}}$, there are only two distinct SPT phases: the product state $|+ + \cdots +\rangle$ and the cluster state. One way to distinguish these two phases is to gauge the $\mathbb{Z}_2^{\mathrm{e}} \times \mathbb{Z}_2^{\mathrm{o}}$ symmetry. Gauging the symmetry via minimal coupling amounts to doing Kramers-Wannier transformations on even and odd sites separately. Up to a lattice translation, this is the same as (4). The Hamiltonians of these two phases after gauging are

$$\begin{aligned}H'_{\mathrm{trivial}} &= -\sum_{j=1}^{L} Z'_{j-1}Z'_{j+1},\\ H'_{\mathrm{cluster}} &= -\sum_{j=1}^{L} Z'_{j-1}X'_jZ'_{j+1}.\end{aligned} \tag{9}$$

The two SPT phases are distinct in that they are mapped to a symmetry breaking phase and a symmetry preserving phase under gauging. In particular, the cluster Hamiltonian is invariant under gauging.

For later convenience, we will choose to distinguish these two SPT phases by a twisted gauging. In the condensed matter literature, it is known as the Kennedy-Tasaki (KT) transformation [33,34] (see also [22,35]):

$$X_j \rightsquigarrow \hat{X}_j, \qquad Z_{j-1}Z_{j+1} \rightsquigarrow \hat{Z}_{j-1}\hat{X}_j\hat{Z}_{j+1}. \tag{10}$$

(In the field theory context, the untwisted and twisted gaugings are respectively referred to as the **S** and **TST** gaugings, representing different elements of the modular group.) The two SPT phases after twisted gauging become

$$\begin{aligned}\hat{H}_{\mathrm{trivial}} &= -\sum_j \hat{X}_j,\\ \hat{H}_{\mathrm{cluster}} &= -\sum_j \hat{Z}_{j-1}\hat{Z}_{j+1}.\end{aligned} \tag{11}$$

The trivial phase is invariant under twisted gauging, and the cluster state goes to the symmetry breaking phase. See Table I for a summary.

Now we are ready to study the SPT phases of the noninvertible symmetry. As we show in the Supplemental Material [28], after the twisted gauging of $\mathbb{Z}_2^{\mathrm{e}} \times \mathbb{Z}_2^{\mathrm{o}}$, the noninvertible symmetry (8) becomes an anomalous $\hat{\mathbb{Z}}_2^{\mathrm{e}} \times \hat{\mathbb{Z}}_2^{\mathrm{o}} \times \hat{\mathbb{Z}}_2^{\mathsf{V}}$ symmetry generated by

$$\hat{\eta}^{\mathrm{e}} = \prod_{j:\mathrm{even}} \hat{X}_j, \quad \hat{\eta}^{\mathrm{o}} = \prod_{j:\mathrm{odd}} \hat{X}_j, \quad \hat{\mathsf{V}} = \prod_j \hat{\mathsf{CZ}}_{j,j+1}. \tag{12}$$

TABLE I. The KT transformation maps the product state and the cluster state to a symmetry preserving phase and a symmetry breaking phase, respectively.

$\mathbb{Z}_2^{\mathrm{e}} \times \mathbb{Z}_2^{\mathrm{o}}$ SPT states		Symmetry breaking pattern of $\hat{\mathbb{Z}}_2^{\mathrm{e}} \times \hat{\mathbb{Z}}_2^{\mathrm{o}}$
$\lvert+ + \cdots +\rangle$	$\xrightarrow[\mathrm{KT}]{\mathbf{TST}}$	Unbroken
$\lvert\mathrm{cluster}\rangle$		Completely broken

In other words, applying the transformation (10) to any Hamiltonian invariant under (8) results in a Hamiltonian with the $\hat{\mathbb{Z}}_2^{\mathrm{e}} \times \hat{\mathbb{Z}}_2^{\mathrm{o}} \times \hat{\mathbb{Z}}_2^{\mathsf{V}}$ symmetry. The 't Hooft anomaly of this symmetry is described by the $2+1$d invertible field theory $(-1)^{\int \hat{A}^{\mathrm{e}} \cup \hat{A}^{\mathrm{o}} \cup \hat{A}^{\mathsf{V}}}$, known as the type III anomaly [36,37]. (Here $\hat{A}$'s are discrete background gauge fields.) We summarize this relation as

$$\mathrm{Rep}(\mathrm{D}_8) \xrightarrow[\mathrm{KT}]{\mathbf{TST}} \underbrace{\hat{\mathbb{Z}}_2^{\mathrm{e}} \times \hat{\mathbb{Z}}_2^{\mathrm{o}} \times \hat{\mathbb{Z}}_2^{\mathsf{V}}}_{\text{type III anomaly}} \tag{13}$$

The three noninvertible SPT phases can be distinguished by the symmetry breaking patterns of this dual symmetry after the KT transformation.

Given any of the three $\mathrm{Rep}(\mathrm{D}_8)$ SPT phases, the fact that it has the noninvertible symmetry D implies the invariance under the untwisted gauging of $\mathbb{Z}_2^{\mathrm{e}} \times \mathbb{Z}_2^{\mathrm{o}}$ [24,38]. As discussed above, the cluster state is invariant under the untwisted gauging, but the product state is not. Therefore, any $\mathrm{Rep}(\mathrm{D}_8)$ SPT must be in the same phase as the cluster state as far as the $\mathbb{Z}_2^{\mathrm{e}} \times \mathbb{Z}_2^{\mathrm{o}}$ symmetry is concerned.

This implies that after the twisted gauging, the dual $\hat{\mathbb{Z}}_2^{\mathrm{e}} \times \hat{\mathbb{Z}}_2^{\mathrm{o}}$ symmetry must be completely broken, with a single $\mathbb{Z}_2$ subgroup of $\hat{\mathbb{Z}}_2^{\mathrm{e}} \times \hat{\mathbb{Z}}_2^{\mathrm{o}} \times \hat{\mathbb{Z}}_2^{\mathsf{V}}$ preserved.(The entire $\hat{\mathbb{Z}}_2^{\mathrm{e}} \times \hat{\mathbb{Z}}_2^{\mathrm{o}} \times \hat{\mathbb{Z}}_2^{\mathsf{V}}$ cannot be completely broken. This would have led to eightfold degenerate ground states which cannot arise from gauging $\mathbb{Z}_2 \times \mathbb{Z}_2$ of a model with a single gapped ground state.) *A priori*, there are four options for this unbroken $\mathbb{Z}_2$ subgroup, generated by $\hat{\mathsf{V}}$, $\hat{\mathsf{V}}\hat{\eta}^{\mathrm{e}}$, $\hat{\mathsf{V}}\hat{\eta}^{\mathrm{o}}$, or $\hat{\mathsf{V}}\hat{\eta}^{\mathrm{e}}\hat{\eta}^{\mathrm{o}}$. However, the diagonal subgroup generated by $\hat{\mathsf{V}}\hat{\eta}^{\mathrm{e}}\hat{\eta}^{\mathrm{o}}$ cannot be preserved since it is anomalous, which is obtained from $(-1)^{\int \hat{A}^{\mathrm{e}} \cup \hat{A}^{\mathrm{o}} \cup \hat{A}^{\mathsf{V}}}$ by setting $\hat{A}^{\mathrm{e}} = \hat{A}^{\mathrm{o}} = \hat{A}^{\mathsf{V}}$. This leaves us with three possible symmetry breaking patterns in Table II.

$\hat{\mathbb{Z}}_2^{\mathsf{V}}$ preserving phase: The original cluster Hamiltonian after gauging is given by (11). The order parameters of the gauged theory are given by $\hat{Z}_0$ and $\hat{Z}_1$ which are both invariant under $\hat{\mathsf{V}}$. Thus the cluster state preserves the $\hat{\mathbb{Z}}_2^{\mathsf{V}}$ subgroup.

$\mathrm{diag}(\hat{\mathbb{Z}}_2^{\mathsf{V}} \times \hat{\mathbb{Z}}_2^{\mathrm{e}})$ preserving phase: For the second option, we propose the following Hamiltonian for the gauged system:

TABLE II. The three $\mathrm{Rep}(\mathrm{D}_8)$ SPT phases are distinguished by the symmetry breaking patterns of the dual $\hat{\mathbb{Z}}_2^{\mathrm{e}} \times \hat{\mathbb{Z}}_2^{\mathrm{o}} \times \hat{\mathbb{Z}}_2^{\mathsf{V}}$ symmetry after the KT transformation. For all three phases, $\hat{\mathbb{Z}}_2^{\mathrm{e}} \times \hat{\mathbb{Z}}_2^{\mathrm{o}}$ is spontaneously broken, but the unbroken subgroup is different.

$\mathrm{Rep}(\mathrm{D}_8)$ SPT states		Symmetry breaking pattern of $\hat{\mathbb{Z}}_2^{\mathrm{e}} \times \hat{\mathbb{Z}}_2^{\mathrm{o}} \times \hat{\mathbb{Z}}_2^{\mathsf{V}}$
$\lvert\mathrm{cluster}\rangle$		$\hat{\mathbb{Z}}_2^{\mathsf{V}}$ unbroken
$\lvert\mathrm{odd}\rangle$	$\xrightarrow[\mathrm{KT}]{\mathbf{TST}}$	$\mathrm{diag}(\hat{\mathbb{Z}}_2^{\mathsf{V}} \times \hat{\mathbb{Z}}_2^{\mathrm{e}})$ unbroken
$\lvert\mathrm{even}\rangle$		$\mathrm{diag}(\hat{\mathbb{Z}}_2^{\mathsf{V}} \times \hat{\mathbb{Z}}_2^{\mathrm{o}})$ unbroken

$$\hat{H}_{\mathrm{odd}} = \sum_{n=1}^{L/2} \hat{Z}_{2n-1}\hat{Z}_{2n+1} - \sum_{n=1}^{L/2} \hat{Y}_{2n}\hat{Y}_{2n+2}(1 + \hat{Z}_{2n-1}\hat{Z}_{2n+3}), \tag{14}$$

and take the number of sites L to be a multiple of 4. This is a commuting Pauli Hamiltonian and thus is exactly solvable. (Crucially, the Hamiltonian is frustration free in the sense that even though different terms in the Hamiltonian are not linearly independent, it is consistent to minimize all of them simultaneously.) Note that the projection factor in the second term does not affect the ground states. This is because for the ground states $\hat{Z}_{2n-1}\hat{Z}_{2n+1} = -1$ and thus $\hat{Z}_{2n-1}\hat{Z}_{2n+3} = 1$. The ground space is determined by $L-2$ independent constraints $\hat{Z}_{2n-1}\hat{Z}_{2n+1} = -1$ and $\hat{Y}_{2n}\hat{Y}_{2n+2} = 1$. This leads to a fourfold degeneracy with ground states given by $\hat{Z}_1 = -\hat{Z}_3 = \cdots = -\hat{Z}_{L-1} = \pm 1$ and $\hat{Y}_0 = \hat{Y}_2 = \cdots = \hat{Y}_{L-2} = \pm 1$. The two order parameters can be taken to be $\hat{Z}_1$ and $\hat{Y}_2(1 - \hat{Z}_1\hat{Z}_3)$, which are both invariant under $\hat{\mathsf{V}}\hat{\eta}^{\mathrm{e}}$.

Next, we undo the twisted gauging of $\mathbb{Z}_2^{\mathrm{e}} \times \mathbb{Z}_2^{\mathrm{o}}$ to find the Hamiltonian for this $\mathrm{Rep}(\mathrm{D}_8)$ SPT phase:

$$H_{\mathrm{odd}} = \sum_{n=1}^{L/2} Z_{2n-1}X_{2n}Z_{2n+1} - \sum_{n=1}^{L/2} Y_{2n}X_{2n+1}Y_{2n+2} + \sum_{n=1}^{L/2} Z_{2n-1}Z_{2n}X_{2n+1}Z_{2n+2}Z_{2n+3}. \tag{15}$$

This Hamiltonian has a unique ground state, denoted by $\lvert\mathrm{odd}\rangle$, stabilized by the following L generators:

$$-Z_{2n-1}X_{2n}Z_{2n+1} = 1, \qquad Y_{2n}X_{2n+1}Y_{2n+2} = 1. \tag{16}$$

Note that while $-Z_{2n-1}X_{2n}Z_{2n+1}$ is invariant under D, $Y_{2n}X_{2n+1}Y_{2n+2}$ is mapped to a product of terms in (16). Similarly, the individual terms in the first sum of H_{odd} are invariant under D, while the terms in the second and the third sums are mapped to each other.

Explicitly, the $\lvert\mathrm{odd}\rangle$ state is given by

$$\lvert\mathrm{odd}\rangle = \prod_{n=1}^{L/2} \mathrm{CZ}_{2n-1,2n+1} \prod_{j=1}^{L} \mathrm{CZ}_{j,j+1} \lvert - - \cdots - \rangle. \tag{17}$$

In the Supplemental Material [28], we show $\mathsf{D}\lvert\mathrm{odd}\rangle = 2(-1)^{L/4}\lvert\mathrm{odd}\rangle$, while $\mathsf{D}\lvert\mathrm{cluster}\rangle = 2\lvert\mathrm{cluster}\rangle$.

$\mathrm{diag}(\hat{\mathbb{Z}}_2^{\mathsf{V}} \times \hat{\mathbb{Z}}_2^{\mathrm{o}})$ preserving phase: The third option is related to the previous one by exchanging $\mathrm{e} \leftrightarrow \mathrm{o}$. More precisely, the Hamiltonian for this state, denoted as $\lvert\mathrm{even}\rangle$, is obtained by conjugating H_{odd} with the lattice translation, i.e., $H_{\mathrm{even}} = T H_{\mathrm{odd}} T^{-1}$.

We have thus identified three $\mathrm{Rep}(\mathrm{D}_8)$ SPT phases, $\lvert\mathrm{cluster}\rangle$, $\lvert\mathrm{odd}\rangle$, and $\lvert\mathrm{even}\rangle$. The SPT states $\lvert\mathrm{odd}\rangle$ and $\lvert\mathrm{even}\rangle$ are related to the cluster state by the finite-depth circuits

$$\prod_{j=1}^{L} Z_j \prod_{n=1}^{L/2} \mathrm{CZ}_{2n-1,2n+1} \quad \text{and} \quad \prod_{j=1}^{L} Z_j \prod_{n=1}^{L/2} \mathrm{CZ}_{2n,2n+2}, \tag{18}$$

respectively. The gates of the circuits can be made $\mathbb{Z}_2^{\mathrm{e}} \times \mathbb{Z}_2^{\mathrm{o}}$ symmetric:

$$|\mathrm{odd}\rangle = \prod_{k=1}^{L/4} Z_{4k-2} Z_{4k} \prod_{n=1}^{L/2} e^{\frac{\pi i}{4}(Z_{2n-1}Z_{2n+1}-1)} |\mathrm{cluster}\rangle. \qquad (19)$$

This shows that the three SPT states are in the same $\mathbb{Z}_2^{\mathrm{e}} \times \mathbb{Z}_2^{\mathrm{o}}$ SPT phase. However, they are different SPT phases for $\mathrm{Rep}(\mathrm{D}_8)$. Indeed the circuits in (18) do not commute with the noninvertible symmetry operator D.

Edge modes—In the previous section, we distinguished the three SPT phases by their different symmetry breaking patterns after gauging a part of the noninvertible symmetry. Here, we discuss the edge modes on the boundary of these phases.

For ordinary on site symmetries, a (nontrivial) SPT phase is characterized by the edge modes or the projective algebra of the symmetry operator on an open chain with two boundaries. However, our noninvertible symmetry is not a product of local unitary operators, and it is unclear how to "cut it open." We therefore need a more universal diagnostic for SPT phases, one that is equivalent to the standard one for on site symmetries, but also applicable to symmetries that are not necessarily on site. To this end, we reinterpret an open chain system as an *interface* between the (nontrivial) SPT phase and the product state on a closed chain. The relative difference between this SPT phase and the product state is then diagnosed by the edge modes at the interfaces. The advantage of this simple reinterpretation is that we never have to cut open the symmetry operator, and this criterion can be readily generalized to noninvertible symmetries. See Sec. IV of the Supplemental Material [28], which includes Refs. [38–69], for a detailed discussion.

Here we focus on the interface between the SPT states $|\mathrm{cluster}\rangle$ and $|\mathrm{odd}\rangle$. Consider a closed chain of L sites, where the cluster state is on half of the chain between sites 1 and ℓ with $1 < \ell < L$, and the $|\mathrm{odd}\rangle$ state is in the region between sites ℓ and L. We assume $L - \ell$ to be a multiple of 4 and L to be even. Consider the $\mathrm{Rep}(\mathrm{D}_8)$-symmetric interface Hamiltonian:

$$\begin{aligned} H_{\mathrm{cluster|odd}} &= -\sum_{j=1}^{\ell} Z_{j-1} X_j Z_{j+1} + \sum_{n=\frac{\ell}{2}+1}^{\frac{L}{2}} Z_{2n-1} X_{2n} Z_{2n+1} \\ &\quad - \sum_{n=\frac{\ell}{2}+1}^{\frac{L}{2}-2} Y_{2n} X_{2n+1} Y_{2n+2} (1 + Z_{2n-1} X_{2n} X_{2n+2} Z_{2n+3}). \end{aligned} \qquad (20)$$

The ground space $\mathcal{H}_{\mathrm{cluster|odd}}$ is stabilized by the following $2L-2$ generators:

$$\begin{aligned} Z_{j-1} X_j Z_{j+1} &= 1 \quad \text{for } j = 1, \ldots, \ell, \\ -Z_{2n-1} X_{2n} Z_{2n+1} &= 1 \quad \text{for } n = \ell/2 + 1, \ldots, L/2, \\ Y_{2n} X_{2n+1} Y_{2n+2} &= 1 \quad \text{for } n = \ell/2 + 1, \ldots, L/2 - 2. \end{aligned} \qquad (21)$$

Hence the ground space is fourfold degenerate, signaling edge modes at the interfaces.

Before identifying the edge modes, we first discuss the projective algebra of the noninvertible symmetry at each of the interfaces, which protects the edge modes. The action of the symmetry operators on any state $|\psi\rangle \in \mathcal{H}_{\mathrm{cluster|odd}}$ factorizes into local factors at the interfaces

$$\begin{aligned} \mathsf{D}|\psi\rangle &= (-1)^{\frac{L-\ell}{4}} \left(\mathsf{D}_{\mathrm{L}}^{(1)} \mathsf{D}_{\mathrm{R}}^{(1)} + \mathsf{D}_{\mathrm{L}}^{(2)} \mathsf{D}_{\mathrm{R}}^{(2)} \right) |\psi\rangle, \\ \eta^{\mathrm{o}} |\psi\rangle &= \eta_{\mathrm{L}}^{\mathrm{o}} \eta_{\mathrm{R}}^{\mathrm{o}} |\psi\rangle, \\ \eta^{\mathrm{e}} |\psi\rangle &= |\psi\rangle, \end{aligned} \qquad (22)$$

where

$$\begin{aligned} \mathsf{D}_{\mathrm{L}}^{(1)} &= Y_{L-2} Y_{L-1} Z_L, \qquad \mathsf{D}_{\mathrm{R}}^{(1)} = Z_{\ell+1}, \\ \mathsf{D}_{\mathrm{L}}^{(2)} &= Z_{L-1}, \qquad \mathsf{D}_{\mathrm{R}}^{(2)} = Z_\ell Y_{\ell+1} Y_{\ell+2}, \\ \eta_{\mathrm{L}}^{\mathrm{o}} &= Y_{L-2} X_{L-1} Z_L, \qquad \eta_{\mathrm{R}}^{\mathrm{o}} = Z_\ell X_{\ell+1} Y_{\ell+2}. \end{aligned} \qquad (23)$$

Here L and R stand for the left and right interfaces around sites $j = L$ and $j = \ell$, respectively; see Fig. 1. We find that the *local* factors of D are charged under η^{o} (but not under η^{e}):

$$\eta^{\mathrm{o}} \mathsf{D}_{\mathrm{L}}^{(I)} = -\mathsf{D}_{\mathrm{L}}^{(I)} \eta^{\mathrm{o}}, \quad \eta^{\mathrm{o}} \mathsf{D}_{\mathrm{R}}^{(I)} = -\mathsf{D}_{\mathrm{R}}^{(I)} \eta^{\mathrm{o}}, \quad I = 1, 2. \qquad (24)$$

Therefore, there is a projective algebra at each interface between the local factors:

$$\eta_{\mathrm{L}}^{\mathrm{o}} \mathsf{D}_{\mathrm{L}}^{(I)} = -\mathsf{D}_{\mathrm{L}}^{(I)} \eta_{\mathrm{L}}^{\mathrm{o}}, \quad \eta_{\mathrm{R}}^{\mathrm{o}} \mathsf{D}_{\mathrm{R}}^{(I)} = -\mathsf{D}_{\mathrm{R}}^{(I)} \eta_{\mathrm{R}}^{\mathrm{o}}, \quad I = 1, 2, \qquad (25)$$

whereas $\eta_{\mathrm{L}}^{\mathrm{e}} \mathsf{D}_{\mathrm{L}}^{(I)} = \mathsf{D}_{\mathrm{L}}^{(I)} \eta_{\mathrm{L}}^{\mathrm{e}}$ and $\eta_{\mathrm{R}}^{\mathrm{e}} \mathsf{D}_{\mathrm{R}}^{(I)} = \mathsf{D}_{\mathrm{R}}^{(I)} \eta_{\mathrm{R}}^{\mathrm{e}}$. This projective algebra matches with the continuum discussion in [[27], (3.62)]. Importantly, the *global* $\mathsf{D}, \eta^{\mathrm{e}}, \eta^{\mathrm{o}}$ operators realize the algebra (8) linearly.

On the left interface, the operators $\eta_{\mathrm{L}}^{\mathrm{o}}$ and $\mathsf{D}_{\mathrm{L}}^{(2)}$ form a Pauli algebra acting on a qubit localized around site L, and similarly for the operators $\eta_{\mathrm{R}}^{\mathrm{o}}$ and $\mathsf{D}_{\mathrm{R}}^{(1)}$ at the right interface. These operators commute with the interface Hamiltonian and form a complete basis of operators acting on its ground space. These edge modes are protected by the projective algebra (24) and cannot be lifted by symmetric deformations of the Hamiltonian near the interfaces.

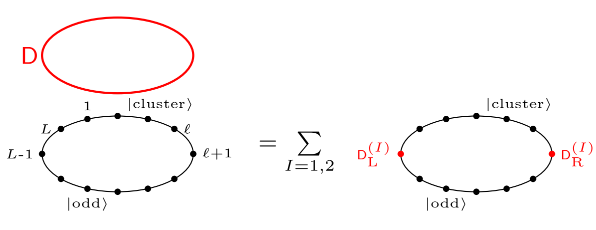

FIG. 1. Localization of the noninvertible operator D at the interfaces between SPT phases. The interface system (20) is locally in the cluster state in the region $j = 1, 2, \ldots, \ell$, and locally in the $|\mathrm{odd}\rangle$ state in the complement region. The operator D factorizes into local factors $\mathsf{D}_{\mathrm{L}}^{(1)} \mathsf{D}_{\mathrm{R}}^{(1)} + \mathsf{D}_{\mathrm{L}}^{(2)} \mathsf{D}_{\mathrm{R}}^{(2)}$ [multiplied by $(-1)^{(L-\ell)/4}$] on the ground space.

Similarly, the interface between $|\text{cluster}\rangle$ and $|\text{even}\rangle$ is obtained by conjugating the previous interface system by the lattice translation T. For this interface, we find that the local factors of D are charged under η^{e}, leading to a projective algebra involving their local factors. We leave the interface between $|\text{even}\rangle$ and $|\text{odd}\rangle$ for the future.

No symmetric entangler for noninvertible SPT phases—For an ordinary invertible symmetry G, without knowing the microscopic details, it is only meaningful to discuss the relative difference between two SPT phases. In continuum field theory, this corresponds to the ambiguity of adding a counterterm at short distances.

In lattice systems with an on site symmetry, one often *declares* that the product state is the trivial SPT state. However, from the *macroscopic* point of view, the notion of trivial SPT is only a choice and can be changed by doing a change of basis in the microscopic variables. Indeed, there is usually a (locality-preserving) unitary operator U that commutes with the symmetry operators and maps the product state to any other SPT state. This operator U is called the *entangler*, since it maps the product state to a short-range entangled state. We refer to it as a symmetric entangler if it commutes with the symmetry operator globally.

Given two G-SPT states, their tensor product state is another SPT state with respect to the diagonal subgroup of $G \times G$. Hence, ordinary G-SPT phases can be multiplied via stacking, which is implemented by a symmetric entangler macroscopically.

However, for noninvertible symmetries, there is no notion of stacking operation for the following reason [27]. Starting with two systems with the same noninvertible symmetry, the tensor product system does not have a "diagonal" noninvertible symmetry of the same type [70]. For instance, taking two systems with symmetries generated by $\eta^{\mathrm{e}}, \eta^{\mathrm{o}}, \mathsf{D}$, the "diagonal" symmetry in the tensor product theory contains $\mathsf{D} \otimes \mathsf{D}$, which obeys a different algebra since $(\mathsf{D} \otimes \mathsf{D})^2 \neq (1 + \eta^{\mathrm{e}} \otimes \eta^{\mathrm{e}})(1 + \eta^{\mathrm{o}} \otimes \eta^{\mathrm{o}})$. Thus, there is no notion of invertibility for SPT phases protected by noninvertible symmetries. (Therefore, the adjective "noninvertible" in "noninvertible symmetry protected topological phase" is attached to the word "symmetry," rather than to the word "phase." This is similar to the use of the term "subsystem symmetry protected topological phase.").

In summary, stacking exists for invertible symmetries, so the notion of a trivial (invertible) SPT phase exists, but not in a canonical way. Indeed, the classification is relative, and declaring the product state as trivial is a *choice*. Stacking does not exist for noninvertible symmetries, so the notion of a trivial noninvertible SPT phase does not exist, even if we specify a microscopic description.

This suggests that there is no $\mathrm{Rep}(\mathrm{D}_8)$-symmetric entangler between the three different SPT states $|\text{cluster}\rangle$, $|\text{odd}\rangle$, $|\text{even}\rangle$. Below we argue that there cannot be a $\mathrm{Rep}(\mathrm{D}_8)$-symmetric unitary operator U such that, say, $\mathsf{U}|\text{cluster}\rangle = |\text{odd}\rangle$. The idea is to show that if such an entangler U exists, then there must exist a unitary operator $\hat{\mathsf{U}}$ in the **TST** gauged/KT transformed system such that it is invariant under the dual symmetry $\hat{\mathbb{Z}}_2^{\mathrm{e}} \times \hat{\mathbb{Z}}_2^{\mathrm{o}} \times \hat{\mathbb{Z}}_2^{\mathsf{V}}$ and maps the two symmetry breaking phases to each other. However, this is impossible since the two SPT phases correspond to two different patterns of symmetry breaking after gauging and hence cannot be related by a symmetric unitary operator.

More specifically, we prove in the Supplemental Material [28] that the existence of such a $\mathrm{Rep}(\mathrm{D}_8)$-symmetric entangler U implies the existence of $\hat{\mathsf{U}}$ satisfying

$$\begin{aligned} \hat{\mathsf{U}}\,\hat{\mathsf{V}} &= \hat{\mathsf{V}}\,\hat{\mathsf{U}}, & \hat{\mathsf{U}}\hat{\eta}^{\mathrm{e}} &= \hat{\eta}^{\mathrm{e}}\hat{\mathsf{U}}, \\ \hat{\mathsf{U}}|\widehat{\text{cluster}}\rangle &= |\widehat{\text{odd}}\rangle, & \hat{\mathsf{U}}\hat{\eta}^{\mathrm{o}} &= \hat{\eta}^{\mathrm{o}}\hat{\mathsf{U}}. \end{aligned} \tag{26}$$

Here, $|\widehat{\text{cluster}}\rangle$ and $|\widehat{\text{odd}}\rangle$ are the symmetric ground states of $\hat{H}_{\text{cluster}}$ and $\hat{H}_{\text{odd}}$ in *finite volume*. In infinite volume, each of these two states splits into four superselection sectors, which break the $\hat{\mathbb{Z}}_2^{\mathrm{e}} \times \hat{\mathbb{Z}}_2^{\mathrm{o}}$ symmetry spontaneously. However, (26) cannot be true since $|\widehat{\text{cluster}}\rangle$ represents a $\hat{\mathbb{Z}}_2^{\mathsf{V}}$-preserving phase, while $|\widehat{\text{odd}}\rangle$ represents a spontaneous $\hat{\mathbb{Z}}_2^{\mathsf{V}}$-breaking phase.

Conclusions and outlook—We find a new noninvertible symmetry D in the standard cluster model H_{cluster}, which, together with the $\mathbb{Z}_2^{\mathrm{e}} \times \mathbb{Z}_2^{\mathrm{o}}$ symmetry, forms a $\mathrm{Rep}(\mathrm{D}_8)$ fusion category. This symmetry leads to two new commuting Pauli Hamiltonians, whose ground states $|\text{odd}\rangle$ and $|\text{even}\rangle$ are SPT states protected by $\mathrm{Rep}(\mathrm{D}_8)$. They are distinguished from the cluster Hamiltonian by their symmetry breaking patterns after the KT transformation/**TST** gauging.

While $|\text{cluster}\rangle$, $|\text{odd}\rangle$, $|\text{even}\rangle$ are in the same SPT phase as far as the $\mathbb{Z}_2^{\mathrm{e}} \times \mathbb{Z}_2^{\mathrm{o}}$ symmetry is concerned, they are distinguished by the noninvertible symmetry. This is reflected in the edge modes from the local projective algebra of the noninvertible symmetry.

Finally, we argue that there is no $\mathrm{Rep}(\mathrm{D}_8)$-symmetric entangler relating these three SPT states. This means that two distinct noninvertible SPT phases correspond to two different continuum field theories at long distances that do not differ merely by a local counterterm.

The $\mathrm{Rep}(\mathrm{D}_8)$ fusion category is one of the simplest (nonanomalous) noninvertible symmetries that admit SPT phases, and we constructed the lattice models for *all* of its possible SPT phases. Our models are complementary to the noninvertible SPT models of [71,72], which employ more general local Hilbert spaces than qubits.

There are several future directions. The key to our construction is that different noninvertible SPT phases are distinguished by the conventional spontaneous symmetry breaking patterns in the gauged systems. This construction can be readily generalized to many other noninvertible symmetries in general dimensions. It would also be interesting to explore the anomaly inflow for noninvertible symmetry by the explicit noninvertible SPT lattice models.

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