Landau-Lifschitz Magnets: Exact Thermodynamics and Transport

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The classical Landau-Lifshitz equation—the simplest model of a ferromagnet—provides an archetypal example for studying transport phenomena. In one-spatial dimension, integrability enables the classification of linear and nonlinear mode spectrum. An exact characterization of finite-temperature thermodynamics and transport has nonetheless remained elusive. We present an exact description of thermodynamic equilibrium states in terms of interacting modes. This is achieved by retrieving the classical Landau-Lifschitz model through the semiclassical limit of the integrable quantum spin-S anisotropic Heisenberg chain at the level of the thermodynamic Bethe ansatz description. In the axial regime, the mode spectrum comprises solitons with unconventional statistics, whereas in the planar regime we find two additional types of modes of radiative and solitonic type. Our framework enables analytical study of unconventional transport properties: as an example we study the finite-temperature spin Drude weight, finding excellent agreement with Monte Carlo simulations.

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Introduction—A quantitative understanding of macroscopic phenomena in interacting many-body systems is a central goal of theoretical and experimental physics. However, strong interactions make perturbative calculations unreliable, and to make progress, one has to identify appropriate collective degrees of freedom. An emblematic example of this paradigm is solitons, referring to stable particlelike coherent field excitations, found across various domains of physics, including shallow water waves [1], gravity [2], cold-atom gases [3–5], magnets [6,7] and others [8–10]. Since the density of excited solitons is highly suppressed at low temperature, it has been suggested that thermodynamic quantities can be accessed by treating the system as a dilute gas, assuming solitons behave as well-separated quasiparticles [11,12]—giving birth to the phenomenological *soliton-gas approach* [13–18]. While initially proposed only as an approximate technique for capturing physics at low temperature, it has been argued in subsequent works that in integrable models the soliton-gas description should provide accurate results even at finite temperature [13], i.e., far from the dilute gas regime.

The inverse scattering method (ISM) [19–21] constitutes a general framework for studying classical integrable partial differential equations. Within ISM, any field configuration that decays to the classical vacuum at spatial infinity can be uniquely decomposed in terms of delocalized radiative modes (often called phonons) and localized waves called solitons; see Fig. 1(a) for a pictorial representation. A major downside of the ISM is that it cannot directly account for configurations with finite energy density that enter thermodynamic ensembles. Recent theoretical works [22–25] in the domain of soliton gases either deal with special initial conditions or construct multisoliton states with a prescribed set of parameters, without explicitly relating them to thermodynamic state functions. The conventional soliton-gas approach to thermodynamics stipulates that the partition sum can be evaluated exactly by performing a sum over the ISM excitation spectrum. Interactions contribute solely by giving an effective length to quasiparticles, capturing the center of mass displacement after a scattering event [26]. However, whether (i) the ISM modes truly provide an (over)complete set of degrees of freedom, and (ii) the nature of their statistical weights, both of which are pivotal for exact computation of thermodynamic properties, have remained open questions.

Most of the progress has so far been achieved in certain special cases, including the models that only feature radiative modes, such as the sinh-Gordon theory and defocusing nonlinear Schrödinger equation [27], or modes with one type of soliton mode with a single degree of freedom such as, for example, the Toda chain [28–31], the KdV equation [32], and the Ablowitz-Ladik model [33,34]. By contrast, generic models that involve both radiation and solitons, and may even comprise multiples species of nonlinear waves, are much more difficult to describe. Despite intensive efforts, the early attempts to obtain an exact soliton-gas description for the sine-Gordon model [35–38] (which has been achieved only recently in [39]) and the Landau-Lifschitz equation [40,41] have not come to fruition. In fact, numerous inconsistencies and controversies [42] hinted that such an approach might suffer from a fundamental conceptual flaw.

In the meantime, rapid advances in the manipulation of quantum matter [43,44] have steered interest toward the study of quantum integrable systems [45–50]. The powerful tools of thermodynamic Bethe ansatz (TBA) [51] and generalized hydrodynamics [52–54] have delivered a myriad of exact results in various equilibrium and nonequilibrium settings, and led to important experimental confirmations [55–59]. Crucially, the TBA formalism for quantum systems—a quantum analog of the soliton-gas approach—is free of nuances that one typically encounters in classical integrable partial differential equations such as, for example, the determination of correct excitation statistics. This motivates the search for an alternative path to thermodynamics of classical integrable systems: taking the semiclassical limit directly at the level of TBA equations [39,60–64] avoids most of the pitfalls of the phenomenological soliton-gas approach.

In this Letter, we describe how to compute exact thermodynamic properties of the classical Landau-Lifschitz (LL) model—a prototypical model for a ferromagnet—and derive hydrodynamic equations governing the evolution on the ballistic scale. This is achieved by regarding the LL model as the large-spin limit of the integrable quantum spin-S chains [65–69]. While semiclassical limits of integrable quantum magnets have in some capacity been addressed previously in the literature [70–74], we here for the first time manage to determine the complete set of modes, alongside their associated statistical weights, relevant for capturing exact thermodynamic properties. Specifically, our description of thermal Gibbs states (or any generalized Gibbs ensembles [75]) is embedded in the standard TBA framework, thus providing the long sought soliton-gas picture. Most remarkably, unlike radiation and solitons that are conventionally associated with Rayleigh-Jeans and Maxwell-Boltzmann statistics, respectively, the modes obey unorthodox statistics.

Our results provide a direct access to many applications in physics. Here, we wish to particularly highlight the recent discovery of spin superdiffusion in integrable quantum [74,76–81] and classical spin chains [82,83], anomalous spin-current statistics [71,84–86], and anomalous types of transport in nonintegrable classical chains [87–89], which have also attracted significant interest in experimental communities [90–93]. Our work offers a theoretical and computational framework for a detailed investigation of nonequilibrium phenomena, and anomalous transport in particular for which, in spite of tremendous advancements, a complete understanding is still lacking.

The model—The classical field theory of a one-dimensional classical Landau-Lifschitz magnet is governed by the Hamiltonian

$$H_{\mathrm{LL}} = \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}x\{(\partial_x \mathbf{S})^2 + \Delta[1-(S^3)^2]\}, \qquad (1)$$

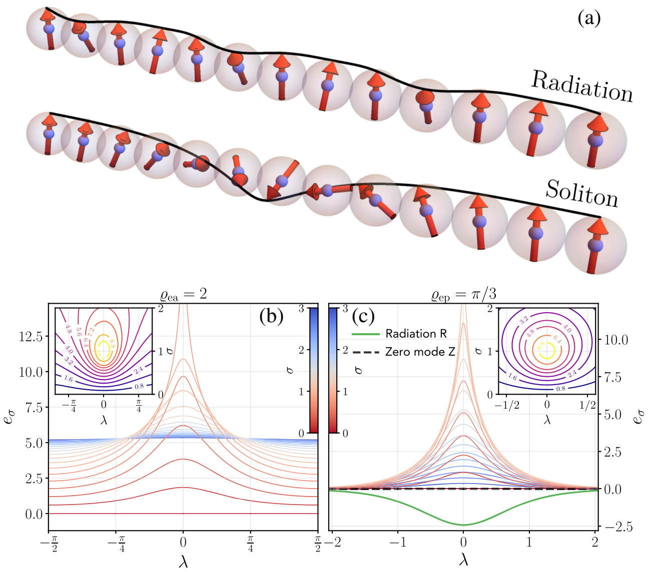

FIG. 1. Landau-Lifschitz excitations. (a) The inverse scattering method decomposes a spin configuration in terms of delocalized radiation (akin to spin waves) and localized solitons as the fundamental excitations above the ferromagnetic vacuum. In the thermodynamic limit in which the mode structure depends on the regime, we focus on the lattice discretization (2). (b) The easy-axis regime supports a continuous family of solitons whose energy $e_\sigma(\lambda)$ is sharply peaked at $(\lambda, \sigma) = (0, 1)$ (see text for details). (c) The easy-plane regime supports two additional types of modes: radiation R with negative energy and a zero mode Z with vanishing energy. Insets of panels (b),(c) show energy contours in the two regimes.

where the spin field $\mathbf{S}(x) = [S^1(x), S^2(x), S^3(x)]$, normalized to unity $\mathbf{S}\cdot\mathbf{S} = 1$, obeys the Lie-Poisson brackets $\{S^a(x), S^b(y)\} = \mathcal{E}\epsilon_{abc}S^c(x)\delta(x-y)$ and $\Delta \in \mathbb{R}$ is the anisotropy. Here, $\mathcal{E}$ is a free parameter used to set the timescale in the equation of motion $\mathrm{d}\mathbf{S}/\mathrm{d}t = \{\mathbf{S}, H\}$. We pick $\mathcal{E} = 2$, which will be justified later on. The third component of total magnetization, $M^3 \equiv \int \mathrm{d}x S^3(x)$, is conserved in time. In the easy-axis regime $\Delta > 0$, the lowest energy configuration is the ferromagnetic vacuum with constant magnetization $S^3(x) = \pm 1$, while in the easy-plane $\Delta < 0$ regime all spins align in the orthogonal plane with $S^3(x) = 0$.

Hereafter, we consider grand-canonical equilibrium ensembles with partition functions $Z(\beta,\mu) \equiv \int \mathcal{D}[\mathbf{S}] e^{-\beta H + \mu(M^3 - M^3_{\mathrm{vac}})}$ (normalized to $Z(0,0) = 1$), where M^3_{vac} denotes the average of M^3 in the ferromagnetic vacuum. To establish the validity of our analytical description and to showcase its predictive power, we perform several independent checks. By imposing periodic boundary conditions for a system of length L, we first compute the free energy density $f = -\lim_{L\to\infty} L^{-1} \log Z(\beta,\mu)$, encoding the charge averages (and their static correlation functions) such as, for example, the average magnetization $\langle S^3\rangle$. As a paradigmatic probe of transport, we consider dynamical correlation functions of the spin current and compute the associated Drude weight $\mathcal{D} = \lim_{t\to\infty}\int \mathrm{d}x\langle j(x,t)j(0,0)\rangle_{\mathrm{c}}$, where the spin-current density $j(x,t)$ is defined via $\partial_t S^3(x,t) + \partial_x j(x,t) = 0$. Recall that integrable models in general feature finite nonzero Drude weights, attributed to ballistic propagation of quasiparticle excitations [94]. Both probes, namely average magnetization and spin Drude weight, provide nontrivial checks for the completeness of the derived classical TBA equations. Specifically, in the infinite-temperature limit we analytically retrieve the magnetization curve $\langle S^3\rangle = -1/\mu + \coth\mu$ from our TBA [95]. Additionally, we find excellent agreement with numerical simulations at finite temperatures; see Fig. 2 and Supplemental Material (SM) [95] for the Drude weight and magnetization, respectively.

Conveniently, the LL model (1) admits an integrable lattice discretization [20], the lattice Landau-Lifschitz (LLL) model, $H_{\mathrm{LLL}} = -2\sum_\ell \log\Phi_{\ell,\ell+1}$, with

$$\Phi_{\ell,\ell+1} = \Upsilon(S^3_\ell)\Upsilon(S^3_{\ell+1})\left(S^1_\ell S^1_{\ell+1} + S^2_\ell S^2_{\ell+1}\right) + U(1)U\left(\frac{1}{2}(S^3_\ell + S^3_{\ell+1})\right) - U\left(\frac{1}{2}(S^3_\ell - S^3_{\ell+1})\right), \quad (2)$$

with auxiliary functions $U(y) \equiv \cosh(\varrho_{\mathrm{ea}} y)$ and $\Upsilon(y) \equiv \sqrt{(U(1) - U(y))/(1-y^2)}$, and the easy-axis anisotropy parameter $\varrho_{\mathrm{ea}} \in \mathbb{R}_+$. The easy-plane regime is reached by analytic continuation $\varrho_{\mathrm{ea}} \to i\varrho_{\mathrm{ep}}$, $\varrho_{\mathrm{ep}} \in [-\pi,\pi]$. The continuum limit, yielding Eq. (1), is recovered at large wavelengths by introducing the lattice spacing a, expanding $\mathbf{S}_{\ell\pm1} = \mathbf{S}(x) \pm a\partial_x\mathbf{S}(x) + \mathcal{O}(a^2)$, rescaling the interaction as $\varrho_{\mathrm{ea}} = a\sqrt{\Delta}$, and letting $a \to 0$. Since the field theory is accessible as a limit of the lattice Hamiltonian (2), we subsequently focus our considerations on the lattice model,

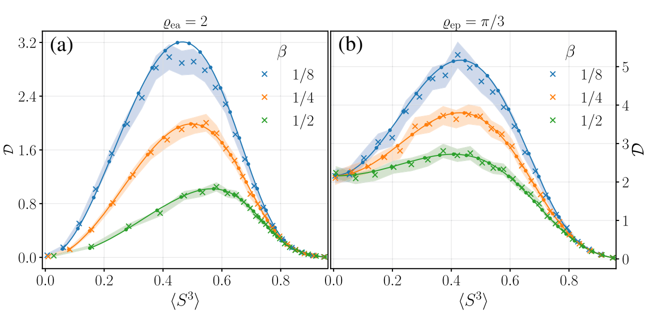

FIG. 2. Spin Drude weight. We consider the lattice Landau-Lifschitz (2) and compare the result of the TBA equations (circles, solid lines as a guide to the eye) with Monte Carlo data (crosses, with shaded regions showing two standard deviation confidence intervals). (a) Easy-axis regime with $\varrho_{\mathrm{ea}} = 2$. (b) Easy-plane regime with $\varrho_{\mathrm{ep}} = \pi/3$. Simulation parameters: system size $L = 2\times10^3$, number of samples $N = 2\times10^3$, time step $\tau = 0.03$; see Ref. [71] for details.

which is also more convenient for performing numerical simulations.

Since the final compact results can be discussed without dwelling on the details of taking the semiclassical limit (which follows previous, albeit simpler, derivations [39,63]), we leave unessential details to the SM [95] and discuss their general aspects and results.

Thermodynamics of integrable models—We first briefly introduce the setting of TBA and define thermodynamic state functions (for more details, see SM [95] or the literature [51]). Individual modes (excitations) are assigned a type (or specie) index "I." The associated (bare) energy and momentum are denoted by $e_I(\lambda)$ and $p_I(\lambda)$, respectively, conveniently parametrized in terms of the rapidity variable λ. In spin chains, a type-I excitation carries m_I quanta of magnetization (relative to the ferromagnetic vacuum). In classical magnets, the two degrees of freedom of classical solitons (e.g., magnetization and momenta) take *continuous* values. For compactness, we introduce an implicit notation for the scalar product $a_I\circ b_I$ and convolution $a_I\star b_I$, evaluated over the rapidity domain for any two quantities a_I and b_I ascribed to specie I, while simultaneously adopting the summation convention in the case of repeated index I (i.e., integration in the case of I having a continuous range; see SM [95] for details).

Gibbs ensembles, or more generally generalized Gibbs ensembles [75], are uniquely identified by the rapidity densities $\rho_I(\lambda)$. The total densities of available states are given by $\rho^t_I = (\kappa_I/2\pi)(\partial_\lambda p_I)^{\mathrm{dr}}$ with $\kappa_I = \mathrm{sign}[\partial_\lambda p_I]$, and represents the effective available phase space for each mode. Owing to interactions, quantities g_I associated to mode I get renormalized. This effect is known as dressing, $g_I \mapsto g^{\mathrm{dr}}_I$, and amounts to solving a *linear* (Fredholm) integral equation of the form $g^{\mathrm{dr}}_I + T_{I,I'}\star(\kappa\vartheta g^{\mathrm{dr}})_{I'} = g_I$, where $T_{I,I'}$ encode the effect of interaction among the species of type I and I' (related to the time delay [20,96,97] induced by scattering), while the filling fractions are given by the ratios $\vartheta_I \equiv \rho_I/\rho^t_I$.

The mode densities ρ_I of the Gibbs state can be inferred by minimizing the free energy $F = \beta\langle H\rangle - \mu(\langle M^3\rangle - M^3_{\mathrm{vac}}) - \mathcal{S}$. Here, $\mathcal{S}$ denotes the thermodynamic entropy, obtained by summing over all modes with appropriate statistical weights $s_I = s_I(\vartheta_I)$. Writing the entropy density as $s \equiv \lim_{L\to\infty}(\mathcal{S}/L) = \rho^t_I\circ s_I$ yields the following spectral resolution of the free energy density [95] $f = \lim_{L\to\infty}(F/L) = (1/2\pi)\kappa_I\partial_\lambda p_I\circ\mathcal{F}_I$, with $\mathcal{F}_I(\vartheta_I) \equiv \vartheta_I s'_I(\vartheta_I) - s_I(\vartheta_I)$. On Gibbs ensembles, the occupations ϑ_I satisfy the following nonlinear integral equations:

$$s'_I(\vartheta_I) = \beta e_I + \mu m_I - T_{I,I'}\star\kappa_{I'}\mathcal{F}_{I'}(\vartheta_{I'}), \quad (3)$$

where $s'_I(\vartheta) = \mathrm{d}s_I(\vartheta)/\mathrm{d}\vartheta$. Once ρ_I have been determined, analytically or numerically, the average charge densities can be simply computed by multiplying them with bare charges and summing over the entire spectrum.

For instance, the magnetization density is given by $\langle S^3\rangle = L^{-1}M_{\rm vac} + m_I \circ \rho_I$. On the other hand, the spin Drude weight assumes the following mode resolution [94,95,98,99]:

$$\mathcal{D} = \rho_I w_I \circ (m_I^{\rm dr} v_I^{\rm eff})^2, \tag{4}$$

where $w_I(\vartheta_I) \equiv -1/[\vartheta_I s_I''(\vartheta_I)]$ accounts for the mode statistics and the effective velocity is $v_I^{\rm eff} \equiv (\partial_\lambda e_I)^{\rm dr}/(\partial_\lambda p_I)^{\rm dr}$. The latter identification is sensitive to the choice of timescale in the equation of motion, and our choice $\mathcal{E} = 2$ ensures its validity; see SM [95].

The functional form of statistical factors s_I discerns the nature of quasiparticle modes: typically in quantum models one finds the Fermi-Dirac statistics $s_{\rm FD}(\vartheta) = -\vartheta \log \vartheta - (1-\vartheta)\log(1-\vartheta)$ [51]. By contrast, in classical systems one usually encounters radiation $s_{\rm Rad}(\vartheta) = \log \vartheta$ [61,62] with Rayleigh-Jeans statistics, or solitons $s_{\rm Sol}(\vartheta) = \vartheta(1 - \log \vartheta)$ [29,30,32] with the associated Boltzmann weight. As detailed out in the remainder of the Letter, the LLL model evades this simple description: we find solitons with unorthodox (renormalized) statistical weights alongside (depending on the regime of anisotropy) radiation and exceptional nondynamical (zero-energy) solitons.

Easy-axis regime—We first inspect the easy-axis regime of the Hamiltonian (2). More details, including the continuum limit (1), can be found in the SM [95]. In this regime, rapidities occupy a compact domain $\lambda \in [-\pi/2, \pi/2]$, and the mode spectrum comprises solely solitons with bare energy $e_\sigma(\lambda)$, labeled by λ and a *continuous* internal variable $\sigma \in \mathbb{R}_+$ associated with magnetization $m_\sigma = 2\sigma$, and with positive parity $\kappa_\sigma = 1$. A typical dispersion law is shown in Fig. 1(b). Sectors with different magnetization signs are disconnected, and the ferromagnetic vacuum reference state must be chosen accordingly [95,100]. The power-law singularity at $(\sigma, \lambda) = (1, 0)$ is a consequence of the logarithmic interaction in Eq. (2). Solitons acquire *renormalized* Boltzmann weights,

$$s_\sigma(\vartheta_\sigma) = s_{\rm Sol}(\vartheta_\sigma) - \sigma^{-2} - \vartheta_\sigma \log \sigma^2. \tag{5}$$

Although the form of s_σ affects the filling fraction (3), we note that the additional terms (constant or linear in ϑ_σ) do not affect the function w_I appearing in the Drude weight (4). The kinematic data retrieved by taking the semiclassical limit is compatible with expressions derived using the ISM (see SM [95] for explicit expressions). In Fig. 2(a) we show the Drude weight obtained by numerically solving the classical TBA equations and independently by performing Monte Carlo simulations [101,102]; see also SM [95]. The divergence of the bare energy $e_\sigma(\lambda)$ at $(\sigma, \lambda) = (1, 0)$; see Fig. 1(b), causes a singularity in the effective velocity $v^{\rm eff}$. The singularity is balanced by a zero of the filling function $\vartheta_\sigma(\lambda) \propto e^{-\beta e_\sigma(\lambda)} \propto [(\sigma-1)^2 \varrho_{\rm ea}^2 + 4\lambda^2]^{2\beta}$, rendering the Drude weight finite. However, this "damping" mechanism diminishes with decreasing β, resulting in a logarithmic divergence of the Drude weight at high temperatures, $\mathcal{D} \propto \log \beta^{-1}$ [95].

Easy-plane regime—Remarkably, in the easy-plane regime there appear *three* distinct types of quasiparticles (above the ferromagnetic vacuum): a continuum of magnetic solitons with renormalized statistics (analogous to those in the easy-axis regime) and $\sigma \in [0, \pi/\varrho_{\rm ep}]$, a single radiative mode R with $m_{\rm R} = 2$ and renormalized entropy $s_{\rm R}(\vartheta_{\rm R}) = s_{\rm Rad}(\vartheta_{\rm R}) + 1 + \log(\varrho_{\rm ep}/\pi)$, and, finally, a special type of soliton mode Z with no bare energy or momentum, $e_{\rm Z} = \partial_\lambda p_{\rm Z} = 0$, characterized by finite magnetization $m_{\rm Z} = 2\pi/\varrho_{\rm ep}$ and renormalized entropy weight $s_{\rm Z}(\vartheta_{\rm Z}) = s_{\rm Sol}(\vartheta_{\rm Z}) + \vartheta_{\rm Z}\log(\varrho_{\rm ep}/\pi)$. Solitons have positive parity $\kappa_\sigma = 1$ and $\kappa_{\rm Z} = 1$, whereas the radiative mode has $\kappa_{\rm R} = -1$. In the planar regime, rapidities span the whole real line, $\lambda \in \mathbb{R}$, while the explicit expressions for dispersion laws and scattering kernels are reported in SM [95]. The dispersion laws of these modes are represented in Fig. 1(c), featuring the same type of singularity as in the easy-axis regime, responsible for a logarithmic divergence of the Drude weight in the high-temperature limit. In Fig. 2(b) we compare the Drude weight (4) with our Monte Carlo numerical data.

The spin-wave limit—Small fluctuations of the ferromagnetic vacuum can be expanded in terms of noninteracting delocalized spin waves, akin to phonons. Such spin waves are not explicitly present in our thermodynamic description and hence should somehow emerge out of the existing set of excitations. The natural expectation is that very extended solitons should be practically indistinguishable from delocalized modes. Indeed, a direct analytic calculation [95] demonstrates that in the easy-axis regime spin waves are retrieved as a cumulative effect of shallow solitons by summing over all σ at fixed rapidity. The same applies for the positive-energy radiation branch in the easy-plane regime. By contrast, the radiative branch with negative energy must be included in the TBA as an independent mode in the easy-plane regime. Note that Z modes carry finite magnetization and are effectively eliminated at low densities.

Discussion—By performing the semiclassical limit of the integrable quantum spin-S chain within the framework of the thermodynamic Bethe ansatz, we obtained the exact thermodynamics and hydrodynamics of the classical Landau-Lifshitz model. The obtained integral equations encoding thermodynamics of the model are consistent with a soliton-gas description, but they display several unexpected features. In the easy-axis regime we find a spectrum of solitons with modified Boltzmann statistics, whereas radiative modes (spin waves) do not appear as independent modes; instead they can be seen as a condensate of wide solitons with small amplitude. The easy-plane regime is even more peculiar: apart from solitons, the spectrum of modes includes localized zero-energy modes and only the negative-energy branch of spin waves.

This Letter brings to the front a number of important questions. The most pressing one concerns the general validity of the soliton-gas approach to thermodynamics, which crucially relies on the stipulated form of statistical weights associated with different types of excitations. Our findings indicate that the anticipated Maxwell-Boltzmann or Rayleigh-Jeans statistics are not always appropriate and thus it is unsafe to *a priori* assume them. Presently we are only able to corroborate this claim through a systematic semiclassical analysis of quantum spectra, while an independent purely classical justification using the tools of inverse scattering [19,20] and finite-gap integration [103–105] is still lacking.

Our results are expected to facilitate further progress in understanding anomalous transport phenomena in integrable magnets and help elucidate the elusive phenomenon of spin superdiffusion at the isotropic point and its intimate connection with the Kadar-Parisi-Zhang to the universality class: while the dynamical exponent and scaling function are by now firmly established, [77,78,80,82,83,106–108], the spin-current fluctuations [85,86,109] reveal a discernibly distinct behavior. Computing the full counting statistics of charge transport [110–112] in the LL magnets using the derived classical TBA equations might lead to important new insights. There are many other interesting questions to address, such as obtaining non-Abelian hydrodynamics that governs the evolution of gauge modes associated to polarization direction of the ferromagnetic vacuum at the isotropic point [113], or the study of thermalization in the presence of integrability breaking [114] with multiple quasiparticle species. Addressing these questions requires both a fully fledged analytical toolbox and extensive numerical benchmarks. The results of our work thus make the classical Landau-Lifschitz model an ideal playground for realizing this program.

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Data availability—Raw data and working codes are available on Zenodo [115].

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