f-Sum Rules for Dissipative Systems

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The f-sum rules set general constraints on the response of a quantum many-body system to an external probe and hold significant relevance in the realm of various spectroscopy measurements. In practical conditions, a system unavoidably couples with the environment and acquires effective dissipation. In this Letter, we derive and prove a set of f-sum rules for dissipative systems. Within the framework of linear response theory, we obtain the system response in both linear order of probe field and dissipation parameter. We formulate and prove one first-order and two second-order dissipative f-sum rules. These rules are validated numerically for some example models, and the realization schemes are proposed. In addition, the potential applications are discussed for two interacting many-body systems.

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Linear response theory occupies a pivotal position in the study of quantum matters. It conveys the idea that the susceptibility of a system can be determined by applying a weak external field and measuring its subsequent response [1]. In condensed matter physics, this concept is extensively used in the understanding of magnetic susceptibility, electrical conductivity, and optical spectroscopy of nearly all kinds of systems [2]. As a notable example, based on linear response theory one can obtain the well-known Kubo formula, which links the electrical conductivity to the Lehmann spectral representation of response correlation function [3,4]. Despite the usually complicated dynamical response of a quantum system, universal relations known as f-sum rules can establish a connection between the response correlation function to conserved quantities of the underlying system. For electron spectroscopy, the f-sum rule states that the frequency integral of conductivity is constrained by the total density of electrons [5,6]. Recently, f-sum rule has been generalized for nonlinear conductivity [7], and optical spectrum [8] and spin correlation functions of cold atomic gases [9]. These constraints play a unique and irreplaceable role in explaining experimental results and validating theoretical and numerical treatments, especially in systems with strong interaction.

On the other hand, real physical systems are typically coupled to their surrounding environment. The semiclassical motion of an open system can be described by the Langevin equation, while the quantum evolution of density matrix is governed by the master equation [10]. To facilitate a simpler and more intuitive understanding, effective non-Hermitian Hamiltonians are adopted when the quantum jump operator terms can be neglected. Recently, much effort has been devoted to the study of non-Hermitian systems and leads to the discovery of many interesting phenomena, such as exceptional points [11], skin effect [12], and non-Hermitian topological states [13–16]. The linear response theory has been extended into the realm of open systems [17]. Meanwhile, dissipative response theory has been proposed as a new tool for analyzing correlated dissipative dynamics and measuring critical properties that are difficult to access in traditional spectroscopy studies [18,19]. As an extension of linear response theory, this method is supported by the experiments on dissipative Bose Hubbard model [20] and benchmarked by recent experiment on dissipative Luttinger liquid [21].

Here, we formulate and prove a set of f-sum rules for a dissipative system, which is coupled to a Markovian environment with an effective dissipation parameter γ and probed by an external classical field $A(t)$. The response of the system is within the linear response theory. While the contribution involving only A satisfies the conventional f-sum rule, the terms proportional to both A and γ do not in general acquire time-translational invariance but have double frequency dependence. By connecting the response correlation function and higher-order derivatives of the time-dependent evolution of an observable, we obtain one first-order and two second-order dissipative f-sum rules, which contain double summation over frequencies. These new rules were applied to three models as examples.

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From these applications, it is evident that the rules can facilitate the measurement of certain physical quantities that are previously unreachable. Additionally, possible experimental implementations are proposed.

We consider a quantum system $\hat{H}_S$ coupled to an environment $\hat{H}_E$ with particle and/or energy exchange. The coupling is assumed to be weak and written in a linear form $\hat{H}_{SE} = \hat{L}^\dagger \hat{\xi} + \hat{L}\hat{\xi}^\dagger$, where $\hat{L}$ and $\hat{\xi}$ are operators of the system and environment, respectively. The environment is assumed to be Markovian, such that the operator $\hat{\xi}$ obeys

$$\begin{aligned}\langle \hat{\xi}(t)\hat{\xi}^\dagger(t')\rangle_E &= 2\gamma\delta(t-t'),\\ \langle \hat{\xi}^\dagger(t)\hat{\xi}(t')\rangle_E &= \langle \hat{\xi}(t)\hat{\xi}(t')\rangle_E = \langle \hat{\xi}^\dagger(t)\hat{\xi}^\dagger(t')\rangle_E = 0,\\ \langle \hat{\xi}^\dagger(t)\rangle_E &= \langle \hat{\xi}(t)\rangle_E = 0, \end{aligned} \tag{1}$$

at low temperature limit. Here, the bracket $\langle\cdot\rangle_E \equiv \mathrm{Tr}[e^{-\beta\hat{H}_E}\cdot]/\mathrm{Tr}[e^{-\beta\hat{H}_E}]$ denotes the trace over environment, $\beta = 1/(k_B T)$ is the inverse temperature with k_B the Boltzmann constant, $\hat{\xi}(t) = e^{i\hat{H}_E t}\hat{\xi}e^{-i\hat{H}_E t}$ is the operator in Heisenberg picture, $\delta(t)$ is the Dirac delta function, and γ is the loss rate [10,22].

Measurements on the system $\hat{H}_S$ are facilitated through the application of an external probing field $A(t)$ by means of the Hamiltonian function $\hat{H}_c(t) = A(t)\hat{O}$, which is employed to monitor the physical observable $\hat{O}$. For simplicity, we assume in the forthcoming discussion that $\hat{O}$ is Hermitian and $A(t) = A\cos(\omega_f t)$, with A the external field amplitude and ω_f the frequency. As we provide in Supplemental Material (SM) [23], the choice of cosine function can facilitate the derivation of dissipative f-sum rules. However, the proof of these rules does not require any specific form of $A(t)$ and is valid for all cases. When subjected to the external field, the system is taken out of equilibrium, and the dynamics is governed by the Hamiltonian $\hat{H}_S + \hat{H}_E + \hat{H}_{SE} + \hat{H}_c(t)$. Accordingly, the expectation value of $\hat{O}$ responds to the classical field $A(t)$. By denoting $\hat{H}_0 = \hat{H}_S + \hat{H}_E$ and considering both $\hat{H}_{SE} + \hat{H}_c(t)$ as perturbation, we can write the response $\delta O(t)$ in the interaction picture as

$$\delta O(t) = \langle \hat{U}^\dagger(t)\hat{O}(t)\hat{U}(t)\rangle - \langle \hat{O}(t)\rangle, \tag{2}$$

where $\hat{O}(t) \equiv e^{i\hat{H}_0 t}\hat{O}e^{-i\hat{H}_0 t} = e^{i\hat{H}_S t}\hat{O}e^{-i\hat{H}_S t}$, the evolution operator $\hat{U}(t) = \hat{\mathcal{T}}e^{-i\int_0^t dt'\hat{V}(t')}$ with $\hat{\mathcal{T}}$ the time-ordering operator, $\hat{V}(t) = e^{i\hat{H}_0 t}[\hat{H}_{SE}(t) + \hat{H}_c]e^{-i\hat{H}_0 t}$. Notice that the expectation value $\langle\cdot\rangle$ is defined with respect to the initial state as $\mathrm{Tr}[e^{-\beta\hat{H}_0}\cdot]/\mathrm{Tr}[e^{-\beta\hat{H}_0}]$.

The evolution operator can be expanded in orders as [5]

$$\hat{U}(t) = 1 - i\int_0^t dt'\hat{V}(t') - \int_0^t dt'\int_0^{t'} dt''\hat{V}(t')\hat{V}(t'') + \cdots. \tag{3}$$

By substituting Eq. (3) into Eq. (2), we derive the complete response $\delta O(t)$. In the scenario where the field amplitude A is small, the term in the linear order of A would dominate in a short duration. Besides, the dissipation also affects the response, with leading effect governed by the terms in linear order of decay rate γ. With that, we can write the response as $\delta\tilde{O}^A(t) = \delta O^A(t) + \delta O^{A\gamma}(t)$, where the first term denotes the linear response of a nondissipative Hermitian system to the probe field A, while the second term is the correction induced by dissipation Hamiltonian $\hat{H}_{SE}$ up to linear order of γ. Notice that when calculating the expectation of $\hat{O}$, there is also a term $\delta O^\gamma(t)$ in linear order of γ. However, this term describes the effect induced by dissipation only and is present even in the absence of probe field, it should not be considered when calculating the response.

Compared to Hermitian systems [23], a dissipative system does not in general hold time translational symmetry and the response correlation function will acquire two independent temporal variables, each corresponding to an independent frequency. By substituting Eq. (3) into Eq. (2), and keeping the terms in order of $A\gamma$, we obtain

$$\delta O^{A\gamma}(t) = \int_0^t dt' A(t')\chi^{A\gamma}(t,t'), \tag{4}$$

where the dissipative response correlation function

$$\begin{aligned}\chi^{A\gamma}(t,t') = {}& i\gamma\int_0^{t'} dt''\big\langle\big\{\big[\hat{O}(t),\hat{O}(t')\big],(\hat{L}^\dagger\hat{L})(t'')\big\} - 2\hat{L}^\dagger(t'')\big[\hat{O}(t),\hat{O}(t')\big]\hat{L}(t'')\big\rangle\\ &+ i\gamma\int_{t'}^{t} dt''\big\langle\big[\big\{\hat{O}(t),(\hat{L}^\dagger\hat{L})(t'')\big\} - 2\hat{L}^\dagger(t'')\hat{O}(t)\hat{L}(t''),\hat{O}(t')\big]\big\rangle.\end{aligned} \tag{5}$$

Here, $\{\cdot,\cdot\}$ denotes anticommutation operation and $\langle\cdot\rangle$ is defined with respect to the initial state as $\mathrm{Tr}[e^{-\beta\hat{H}_S}\cdot]/\mathrm{Tr}[e^{-\beta\hat{H}_S}]$. Although the second line of Eq. (5) always acquires time translational symmetry, the first line does not in general. As a result, one cannot use $\chi^{A\gamma}(t-t')$ and its Fourier transformation to represent the most general $\chi^{A\gamma}(t,t')$. The detailed derivation and symmetry analysis of $\chi^{A\gamma}(t,t')$ are given in SM [23].

With the method provided in SM [23], we can derive and prove the following dissipative f-sum rules

$$\mathcal{F}_1 \equiv \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \omega \chi^{A\gamma}(\omega, \omega') = -\gamma \langle [\mathcal{L}\hat{O}, \hat{O}] \rangle, \tag{6}$$

$$\mathcal{F}_2 \equiv \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \omega^2 \chi^{A\gamma}(\omega, \omega') = -\gamma \langle [\mathcal{L}[\hat{O}, \hat{H}_S], \hat{O}] \rangle - \gamma \langle [[\mathcal{L}\hat{O}, \hat{H}_S], \hat{O}] \rangle, \tag{7}$$

$$\mathcal{F}_3 \equiv \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \omega(\omega + \omega') \chi^{A\gamma}(\omega, \omega') = -\gamma \langle \mathcal{L}[[\hat{O}, \hat{H}_S], \hat{O}] \rangle, \tag{8}$$

where the Lindblad-like operator $\mathcal{L}\hat{X} \equiv \{\hat{X}, \hat{L}^\dagger \hat{L}\} - 2\hat{L}^\dagger \hat{X} \hat{L}$. The first-order rule Eq. (6) is the expectation value of the commutator between $\mathcal{L}\hat{O}$ and $\hat{O}$. If the condition $\langle [\mathcal{L}\hat{O}, \hat{O}] \rangle = 0$ can be established for some circumstances, one can immediately conclude that the system response always satisfies the first conventional f-sum rule [23], regardless of the presence of dissipation. A conspicuous example is when $[\hat{O}, \hat{L}] = [\hat{O}, \hat{L}^\dagger] = 0$. Equations (7) and (8) are the second-order dissipative f-sum rules. It is evident that even when $\hat{O}$ commutes with $\hat{H}_S$, Eq. (7) is nonzero due to the presence of $\langle [[\mathcal{L}\hat{O}, \hat{H}_S], \hat{O}] \rangle$. Thus, it emerges as a unique signature of dissipative systems, in contrast to the Hermitian case where the even-order f-sum rules remain constantly zero. Another distinct scenario arises when $[\hat{L}^\dagger, \hat{L}] = [\hat{L}^\dagger, \hat{H}_S] = [\hat{L}, \hat{H}_S] = 0$ is fulfilled, such that the right-hand side of Eq. (8) is reduced to zero. In this case, the response function $\chi^{A\gamma}(t, t')$ exhibits time translational invariance [23]. Finally, we notice that Eq. (8) acquires a cross frequency term $\omega\omega'$ in the integrand, which is a direct consequence of the broken of time translation invariance. In this sense, only in this rule where the full information of $\chi^{A\gamma}(t, t')$ is characterized, whereas the other two rules of Eqs. (6) and (7) manifest solely the behavior of $\chi^{A\gamma}(t, 0)$.

We emphasize that the dissipative f-sum rules mentioned above pertain exclusively to Markovian environments with Dirac delta shaped autocorrelation functions $\langle \hat{\xi}(t) \hat{\xi}^\dagger(t') \rangle_E$. On the other hand, deriving the f-sum rule from the response correlation function $\chi^{A\gamma}(t, t')$ in Eq. (5) requires only the application of the Heisenberg equation of motion, without depending on the specific form of $\hat{H}_S$. As such, the dissipative f-sum rule can be applied to a wide range of models, especially the ones with various forms of interactions.

Next, we discuss three examples to show the application of sum rules Eqs. (6)–(8). The first example is a double-mode model involving a mechanical oscillator coupled to an optical cavity, as being widely applied in quantum information [30] and quantum motors [31]. The Hamiltonian reads

$$\hat{H}_S = \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \omega_2 \hat{a}_2^\dagger \hat{a}_2 + g(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger), \tag{9}$$

where two different bosonic modes $\hat{a}_1$ and $\hat{a}_2$ with respective frequencies ω_1 and ω_2 are coupled with intensity g. For simplicity, we assume the two modes are in resonance with $\omega_1 = \omega_2 = \omega_0$. For the probing scheme, an external field in the pulse form $A(t) = A\delta(t - t_1)$ is assumed to interact with the position operator of one mode, specified without loss of generality as $\hat{O} = \hat{a}_2 + \hat{a}_2^\dagger$, and the environmental coupling is applied to operator $\hat{L} = \hat{a}_1$, which describes a loss of particles in mode 1. By substituting $\hat{O}$ and $\hat{L}$ into Eqs. (6)–(8), we obtain the dissipative f-sum rules $\mathcal{F}_1 = 0$, $\mathcal{F}_2 = 0$, $\mathcal{F}_3 = 0$.

Although the results are all trivially zero, this simple example is instructive to explain how to measure the dissipative f-sum rules in experiment. To this aim, we need the response correlation function in frequency domain $\chi^{A\gamma}(\omega, \omega')$, which can be obtained from its temporal counterpart $\chi^{A\gamma}(t, t')$ via Fourier transformation. In a practical experiment, one prepares the system in a specific initial state at $t = 0$, and lets it evolve under the dissipative Hamiltonian. Then a probe field is applied at some time $t_1 > 0$, and the system response is measured for $t > t_1$. The experimental sequence is schematically shown in Fig. 1(a), where a meaningful measurement can only exist in the shaded region with $t_1 > 0$ (probe after initialize) and $t > t_1$ (measure after probe). Repeating this protocol with different t_1, we can measure the system response in the shaded area of the two-dimensional temporal space [Fig. 1(b)], where the response function $\chi^{A\gamma}(t, t')$ is of physical meaning. To conduct the double Fourier transform, we need to extend the domain of $\chi^{A\gamma}(t, t')$ to the unphysical region, i.e., the white area in Fig. 1(b). For that, one can employ the expression of Eq. (5) and assign the calculated results to $\chi^{A\gamma}(t, t')$.

For demonstration purposes, we assume the system is prepared initially at $|\psi(0)\rangle = (\sqrt{2}/2)|1,0\rangle - (\sqrt{2}/2)|0,1\rangle$, which is one of the first excited states of $\hat{H}_S$. The response $\delta O(t)$ to the probe pulse applied at time t_1 can be calculated by numerically evolving the Lindblad equation [23]. Two typical results for $t_1 = 0$ and $t_1 = 30\pi/\omega_0$ are shown in Figs. 2(a) and 2(b), respectively. Upon subtracting δO^A and δO^γ, we obtain the contribution $\delta O^{A\gamma}$, from which the dissipative response correlation function can be extracted in

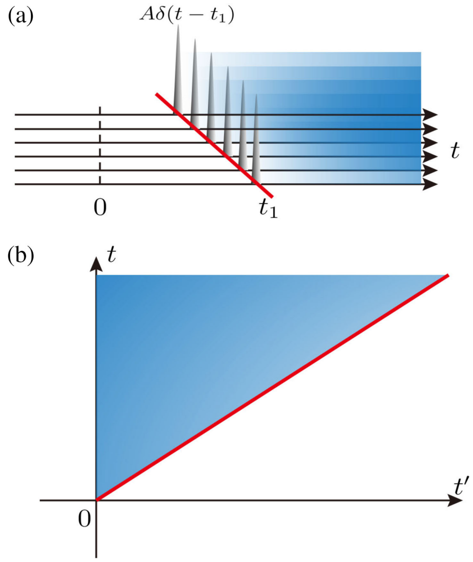

FIG. 1. (a)-(b) Scheme of measurement to validate dissipative f-sum rules.

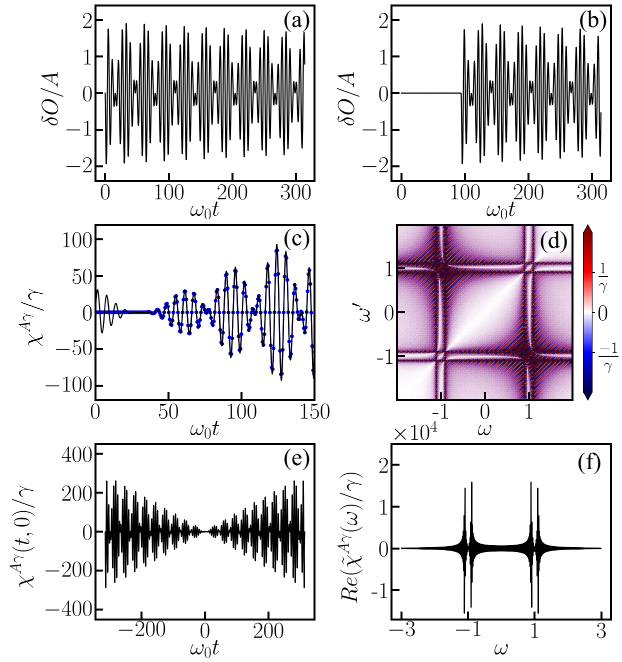

FIG. 2. The numerical results of the double-mode coupled harmonic oscillator model. (a) and (b) Show the evolution of the observable quantity $\delta O/A\gamma$ upon probe pulses applied at $t_1 = 0$ and $t_1 = 30\pi/\omega_0$, respectively. (c) The response correlation function $\chi^{A\gamma}(t, t')$ can be obtained from the measurement of observable. The solid line represents the definition of $\chi^{A\gamma}(t, t' = t_1)$ given in Eq. (5), while the blue points are simulated data captured through the evolution of $\delta O(t)$. (d) The real part of $\chi^{A\gamma}(\omega, \omega')$ with magnitude denoted by false color. (e) Dissipative response correlation function $\chi^{A\gamma}(t, 0)$. The part for $t > 0$ is obtained from numerical simulation and can be measured in the experiment. For the unphysical regime of $t < 0$, an extension is made by either using Eq. (5) or relying on time reflection symmetry. (f) The real part of the Fourier transform of $\chi^{A\gamma}(t, 0)$. Parameters used are $g/\omega_0 = 0.1$ and $\gamma/\omega_0 = A/\omega_0 = 10^{-3}$. The initial state $|\psi(0)\rangle = (\sqrt{2}/2)|1,0\rangle - (\sqrt{2}/2)|0,1\rangle$ is the first excited state of $\hat{H}_S$.

accordance with the choice of pulse shape of probe field, i.e., $\delta O^{A\gamma} = A\Theta(t - t_1)\chi^{A\gamma}(t, t_1)$. In Fig. 2(c), we show $\chi^{A\gamma}(t, t_1 = 10\pi/\omega_0)$ from numerical evolution by blue points as a simulation of what one would measure in an experiment. The data coincide with the prediction of Eq. (5) precisely for $t > t_1$. For the region of $t < t_1$, the response function is zero by definition since the probe has not been applied yet. We then use the results of Eq. (5) to replace the trivial results within that region, as well as the physically undetectable domain of $t < 0$. By applying the same procedure of measurement (for $t > t_1 > 0$) and extension (for $t < t_1$ and $t_1 < 0$) for different t_1, we can obtain the complete set of $\chi^{A\gamma}(t, t')$ and $\chi^{A\gamma}(\omega, \omega')$, as shown in Fig. 2(d). After performing the double-frequency integrations numerically, the f-sum rules can be validated.

Furthermore, we notice that the frequency polynomials in the integrations of $\mathcal{F}_1$ and $\mathcal{F}_2$ are about ω only. This simple dependence suggests that we can analytically carry out the integration over ω' provided that $\chi^{A\gamma}(t, t')$ acquires time translational symmetry with respect to t'. In experiment, this corresponds to the simple case of fixed $t_1 = 0$, i.e., the probe pulse is applied right at the time when the system is initialized. The response correlation function yields $\chi^{A\gamma}(t, 0)$, which is the time invariant part of $\chi^{A\gamma}(t, t')$ [23]. In Figs. 2(e) and 2(f), we show the numerical simulation of $\chi^{A\gamma}(t, 0)$ and its Fourier transform $\tilde{\chi}^{A\gamma}(\omega)$, respectively. Thus, the first two rules $\mathcal{F}_1$ and $\mathcal{F}_2$ can be extracted without using either the complicated double-temporal measurement or the numerical extension of Eq. (5). For this example, all dissipative f-sum rules are satisfied with high precision by using the numerical simulation results with time resolution $0.001\pi/\omega_0 - 0.1\pi/\omega_0$ and cutoff $100\pi/\omega_0$. Consider a system of resonant frequency ~100 MHz, such choice of parameters corresponds to a sampling rate of 30 GSa/s–30 GSa/s–0.3 GSa/s and evolution time of 3 μs, which are both accessible in experiment.

The second example is a generalized Dicke model of N two-level atoms coupled to two cavity fields, which holds paramount importance in the fields of quantum optics and quantum information as it offers a simplified framework for investigating collective interaction phenomena between light and matter, such as superradiance phase transitions [32] and quantum chaos [33,34]. The Hamiltonian reads [35]

$$\hat{H}_S = \omega_a \hat{J}_z + \sum_{n=1,2} \omega_n \hat{a}_n^\dagger \hat{a}_n + g_n(\hat{a}_n^\dagger + \hat{a}_n)\hat{J}_x + U_n \hat{a}_n^\dagger \hat{a}_n \hat{J}_z, \tag{10}$$

where $\hat{a}_{1,2}$ are the cavity field operators with frequency ω_1 and ω_2, respectively, and $\hat{J}_z$ is the collective spin operator of N two-level atoms with the translation frequency ω_a. The cavity fields and atomic collective spin are coupled by both linear term g_n and nonlinear term U_n. For the probing scheme, an external field is assumed to interact with the collective spin operator $\hat{O} = \hat{J}_z$. The dissipation is induced by considering one of the cavity loss, say, $\hat{L} = \hat{a}_1$ with loss rate γ_1. Upon substituting the definition of $\hat{O} = \hat{J}_z$ and $\hat{L}$ into Eqs. (6)–(8), the dissipative f-sum rules are $\mathcal{F}_1 = 0$, $\mathcal{F}_2 = \gamma_1 \langle g_1(\hat{a}_1^\dagger + \hat{a}_1)\hat{J}_x \rangle$, and $\mathcal{F}_3 = \gamma_1 \langle g_1(\hat{a}_1^\dagger + \hat{a}_1)\hat{J}_x \rangle$. Thus, from $\mathcal{F}_2$ alone, we can obtain the linear coupling energy between one of cavity fields and atoms. The distinguishability between two cavities is rooted from the fact that the Lindblad-like operator $\mathcal{L}$ can selectively filter out a specific cavity mode. Moreover, if we measure the atomic spin from another direction with $\hat{O} = \hat{J}_y$, the second dissipative f-sum rule becomes $\mathcal{F}_2 = \gamma_1 \langle g_1(\hat{a}_1^\dagger + \hat{a}_1)\hat{J}_x + 2U_1 \hat{a}_1^\dagger \hat{a}_1 \hat{J}_{qz} \rangle$, which contains both the linear coupling energy and the nonlinear coupling energy amplified by a factor of 2. Thus, by combining with the result of $\hat{J}_z$ measurement, we can separately determine the linear and nonlinear coupling energies for each individual cavity which is not possible for conventional f-sum rule (see more details in SM [23]).

The third example is the Ising-Kondo lattice model

$$\hat{H}_S = -t \sum_{\langle i,j \rangle,\sigma} \hat{c}_{i,\sigma}^\dagger \hat{c}_{j,\sigma} + J \sum_j \hat{s}_j^z \hat{S}_j^z + \Delta \sum_j \hat{S}_j^x, \quad (11)$$

composed by itinerant electrons $\hat{c}_{j,\sigma=\uparrow,\downarrow}$ with nearest-neighbor hopping rate t and localized impurity spins $\hat{S}_j$, which are coupled by on-site Ising-type interaction of strength J. Additionally, the impurities experience a transverse magnetic field Δ. This many-body Hamiltonian is extensively studied in the context of quantum magnetism [36,37] and closely related to many real materials [38,39]. By considering an external field coupled with the impurity spin operator $\hat{S}_n^x$ at a specific site n, and a dissipation induced by the lowering operator $\hat{S}_n^-$ at the same site with loss rate γ, a measurement of the impurity spin $\hat{S}_z^x$ gives f-sum rules $\mathcal{F}_1 = 0$, $\mathcal{F}_2 = \gamma \langle J \hat{s}_n^z \hat{S}_n^z \rangle$, and $\mathcal{F}_3 = \gamma \langle J \hat{s}_n^z (2\hat{S}_n^z + 1) \rangle$. While $\mathcal{F}_2$ provides the on-site Kondo coupling energy, $\mathcal{F}_3$ is mixed by the same Kondo coupling energy and the local electron spin. Combining these two rules we can determine the Hamiltonian parameter J with the aid of local electronic spin measurement.

In this work, we generalize the f-sum rules to dissipative systems. Within the linear response theory, we derive the expectation value of an arbitrary observable $\hat{O}$ of a system subjected to an external driving field and dissipative coupling with a Markovian environment. Up to linear order of both the probe field intensity A and the dissipative coefficient γ, the response function is composed of one term proportional to A denoted by $\hat{O}^A$, and another term proportional to $A\gamma$ denoted by $\hat{O}^{A\gamma}$. While the $\hat{O}^A$ part satisfies the conventional f-sum rules for Hermitian systems, we derive and prove one first-order and two second-order dissipative f-sum rules, which connect the frequency summation of the response correlation function and some specific commutators of the system Hamiltonian. Then we demonstrate the validity and application of the three dissipative f-sum rules for three models as examples and discuss possible experimental realizations. Furthermore, the extension of the dissipative f-sum rule in non-Markovian environments is indeed a worthy endeavor.

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[1] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation and Superfluidity* (Oxford University Press, New York, 2016), Chap. 7.
[2] N. W. Ashcroft and N. D. Mermin, *Solid State Physics*, 1st ed. (Saunders College Publishing, New York, 1976).
[3] R. Kubo, Can. J. Phys. **34**, 1274 (1956).
[4] R. Kubo, J. Phys. Soc. Jpn. **12**, 570 (1957).
[5] G. D. Mahan, *Many-Particle Physics*, 3rd ed. (Kluwer Academic, New York, 2000).
[6] S. Savasta, O. D. Stefano, and F. Nori, Nanophotonics **10**, 465 (2021).
[7] H. Watanabe and M. Oshikawa, Phys. Rev. B **102**, 165137 (2020).
[8] M. O. Oktel, T. C. Killian, D. Kleppner, and L. S. Levitov, Phys. Rev. A **65**, 033617 (2002).
[9] Z. Yu and G. Baym, Phys. Rev. A **73**, 063601 (2006).
[10] R. Zwanzig, *Nonequilibrium Statistical Mechanics*, 1st ed. (Oxford University Press, New York, 2001).
[11] M. De Carlo, F. De Leonardis, R. A. Soref, L. Colatorti, and V. M. N. Passaro, Sensors **22**, 3977 (2022).
[12] X. Zhang, T. Zhang, M.-H. Lu, and Y.-F. Chen, Adv. Phys. X **7**, 2109431 (2022).
[13] S. Yao and Z. Wang, Phys. Rev. Lett. **121**, 086803 (2018).
[14] Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, Phys. Rev. X **8**, 031079 (2018).
[15] B. Zhu, R. Lü, and S. Chen, Phys. Rev. A **89**, 062102 (2014).
[16] V. M. Martinez Alvarez, J. E. Barrios Vargas, M. Berdakin, and L. E. F. Foa Torres, Eur. Phys. J. Spec. Top. **227**, 1295 (2018).
[17] M. Ban, S. Kitajima, T. Arimitsu, and F. Shibata, Phys. Rev. A **95**, 022126 (2017).
[18] L. Pan, X. Chen, Y. Chen, and Z. Hui, Nat. Phys. **16**, 767 (2020).
[19] Y. Chen, J. High Energy Phys. 04 (2021) 215.
[20] R. Bouganne, M. Bosch Aguilera, A. Ghermaoui, J. Beugnon, and F. Gerbier, Nat. Phys. **16**, 21 (2020).
[21] Y. Zhao, Y. Tian, J. Ye, Y. Wu, Z. Zhao, Z. Chi, T. Tian, H. Yao, J. Hu, Y. Chen, and W. Chen, arXiv:2309.10257.

[22] A. Kamenev, *Field Theory of Non-Equilibrium Systems* (Cambridge University Press, Cambridge, England, 2011), Chap. 4.
[23] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.133.100401 for detailed information on the discussion of time translation symmetry of the response correlation function, derivation of the conventional and dissipative f-sum rules, derivation of the Lindblad Equations, and more models as examples with possible experimental implementations, which includes Refs. [24–29].
[24] J. Rammer, *Quantum Field Theory of Non-Equilibrium States*, 1st ed. (Cambridge University Press, Cambridge, England, 2007), Chap. 6.
[25] S. Gupta, Z. Hadzibabic, M. W. Zwierlein, C. A. Stan, K. Dieckmann, C. H. Schunck, E. G. M. van Kempen, B. J. Verhaar, and W. Ketterle, Science **300**, 1723 (2003).
[26] M. Bartenstein, A. Altmeyer, S. Riedl, R. Geursen, S. Jochim, C. Chin, J. H. Denschlag, R. Grimm, A. Simoni, E. Tiesinga, C. J. Williams, and P. S. Julienne, Phys. Rev. Lett. **94**, 103201 (2005).
[27] P. Nozieres and P. David, *Theory of Quantum Liquids* (Westview Press, Boulder, CO, 1994), Vol. 2, Chap. 1.
[28] G. Baym, C. J. Pethick, Z. Yu, and M. W. Zwierlein, Phys. Rev. Lett. **99**, 190407 (2007).
[29] Z. Yu, Ph. D. thesis, University of Illinois at Urbana-Champaign, 2008.
[30] G. J. Qiao, H. X. Gao, and H. D. Liu, Sci. Rep. **8**, 15614 (2018).
[31] A. Kurnosov, L. J. Fernández-Alcázar, R. Bustos-Marún, and T. Kottos, Phys. Rev. Appl. **18**, 064041 (2022).
[32] P. Kirton, M. M. Roses, J. Keeling, and E. G. Dalla Torre, Adv. Quantum Technol. **2**, 1800043 (2019).
[33] C. Emary and T. Brandes, Phys. Rev. Lett. **90**, 044101 (2003).
[34] J. Chávez-Carlos, B. López-del-Carpio, M. A. Bastarrachea-Magnani, P. Stránský, S. Lerma-Hernández, L. F. Santos, and J. G. Hirsch, Phys. Rev. Lett. **122**, 024101 (2019).
[35] F. Dimer, B. Estienne, A. S. Parkins, and H. J. Carmichael, Phys. Rev. A **75**, 013804 (2007).
[36] W.-W. Yang, Y. Zhong, and H.-G. Luo, Phys. Rev. B **102**, 195141 (2020).
[37] X. Zhou, J. Fan, and S. Jia, Phys. Rev. B **109**, 195112 (2024).
[38] H. Tsunetsugu, M. Sigrist, and K. Ueda, Rev. Mod. Phys. **69**, 809 (1997).
[39] B. Li, J.-Q. Yan, D. M. Pajerowski, E. Gordon, A.-M. Nedić, Y. Sizyuk, L. Ke, P. P. Orth, D. Vaknin, and R. J. McQueeney, Phys. Rev. Lett. **124**, 167204 (2020).