Celestial Dual for Maximal Helicity Violating Amplitudes

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(Received 1 April 2024; accepted 1 August 2024; published 30 August 2024)

It is shown that a 2D conformal field theory consisting of a central charge c Liouville theory, a chiral level one, rank N Kac-Moody algebra, and a weight $-3/2$ free fermion holographically generate 4D maximal helicity violating tree-level scattering amplitudes. The correlators of this 2D conformal field theory give directly the 4D leaf amplitudes associated to a single hyperbolic slice of flat space. The 4D celestial amplitudes arise in a large-N and semiclassical large-c limit, according to the holographic dictionary, as a translationally invariant combination of leaf amplitudes. A step in the demonstration is showing that the semiclassical limit of Liouville correlators are given by contact 3D anti–de Sitter Witten diagrams.

DOI: 10.1103/PhysRevLett.133.091603

Introduction—A central goal in celestial holography is to construct simple toy models in which 4D scattering amplitudes are holographically realized by 2D conformal field theory (CFT) correlators. An obstacle to this endeavor has been that the translation invariance of 4D amplitudes of massless particles implies that the low-point correlators are distributional, which is not the case for most familiar 2D CFTs. This obstacle has been circumvented by considering a variety of 4D contexts in which translation (but not Lorentz) invariance is broken [1–13] or by shadow or light transforming the external particle states [14–19]. But a direct holographic reconstruction of translationally invariant 4D scattering amplitudes has so far not been obtained.

In this Letter we obtain such a direct reconstruction of 4D maximal helicity violating (MHV) gluon amplitudes from a 2D CFT. Our work relies heavily on a recent refined analysis of the dictionary for flat space holography including a complete accounting of causal light-cone singularities [20,21]. It was shown that the full 4D amplitudes can be expressed as integrals over leaf amplitudes associated to the 3D anti–de Sitter (AdS_3) leaves of a hyperbolic foliation of flat space. Each leaf amplitude has bulk and boundary representations that are exactly those of familiar AdS holography, and the associated "leaf CFTs" have the corresponding familiar 2D singularity structure. They may be regarded as the primary building blocks of celestial holography. A simple formula was derived for the full celestial amplitudes as a combination of leaf amplitudes. At three points, celestial amplitudes are extracted from leaf amplitudes as a pole in the net conformal weight of the external particles. The distributional form of low-point celestial amplitudes, as mandated by translation invariance, then arises from cancellations between subamplitudes. We refer the reader to [20–22] for details.

In this Letter, we consider a nonunitary 2D CFT consisting of a Liouville field ϕ coupled to N weight $(h,\bar{h})=(\frac{1}{2},0)$ real free fermions ψ_j plus a a single weight $(-\frac{3}{2},0)$ free fermion η. The N fermions lead to a level one $\mathrm{SO}(N)$ Kac-Moody current J^a. (We may similarly obtain an $\mathrm{SU}(N)$ gauge group by using complex fermions.) Positive helicity $\mathrm{SO}(N)$ gluons are identified with Kac-Moody currents dressed by Liouville fields. Negative helicity gluons are then obtained by a further dressing with an η bilinear. These are shown to match, in the appropriate limit, the MHV leaf amplitudes computed in [20]. The leaf-to-celestial dictionary [20] then precisely reproduces the known MHV scattering amplitudes, including the Parke-Taylor factor, the momentum-conserving delta function, and the various Θ functions separating different relative causal configurations of the asymptotic gluons. A key step along the way, detailed in the Supplemental Material [23], is the demonstration that semiclassical Liouville correlators are given by scalar contact AdS_3 Witten diagrams.

Duality here is demonstrated to leading (nontrivial) order expansions about certain limits of both the bulk and the boundary. First, from the bulk point of view, local conformal invariance is equivalent to the subleading soft graviton theorem [24], which of course holds only in a theory of gravity. Hence, we do not ever expect an exact celestial CFT dual to a gauge theory. However, if we take a large-N limit of gravity coupled to a gauge theory with a gauge group of rank N, gravity is suppressed. This suggests that leading order large-N gauge theories may have celestial duals that are limits of 2D CFTs. Accordingly, we herein match only the leading-N correlators of the 2D

Kac-Moody currents. Furthermore, we consider only tree-level MHV amplitudes in the bulk, which are matched to the large-c semiclassical limit of the boundary Liouville theory [25].

This work does *not* comprise the sought-after full holographic duality between a 2D CFT and a 4D quantum theory of gravity. Rather we have achieved the more limited goal of finding a duality between limits of a subset of a 2D boundary theory with those of a 4D bulk theory.

We wish to stress that this Letter is an amalgam of several important precursors. The MHV leaf amplitudes appear in a slightly different context in [3]. A very similar fermion system including the η field is in [26] (itself deriving from the prescient work [27]) and our construction was partly inspired by those in twisted holography [1,6]. An important connection to light operators in the semiclassical Liouville was noted in a slightly different context in [5,7,28,29]. Finally, the connection between 2D Liouville and 4D gauge theory found by [30] may be related.

The celestial CFT—In this section, we describe a relatively simple 2D CFT that generates 4D MHV amplitudes. This CFT has three components: the classical ($b \to 0$) limit of Liouville theory, a set of N free fermions ψ^i, and a free chiral fermion η of weight $-3/2$. We will refer to this as the dressed Liouville theory. In the next section, we describe the holographic dictionary through which MHV amplitudes are reproduced in the classical ($b \to 0$, $N \to \infty$) limit of dressed Liouville theory.

Chiral fermion sector: The free fermion operators have the operator product expansions (OPEs), for $i, j = 1, \ldots, N$,

$$\psi^i(z)\psi^j(w) = \frac{\delta^{ij}}{z-w} + :\psi^i\psi^j:(z) + O(z-w)$$
$$\eta(z)\eta(w) = (z-w)\eta\partial\eta(z) + O[(z-w)^2], \tag{1}$$

where ψ^i have weights $(1/2, 0)$ and η has weight $(-3/2, 0)$.

The ψ^i generate a level-1 Kac-Moody current algebra for $\mathrm{SO}(N)$ with currents

$$J^a(z) = \frac{1}{2}T^a_{ij}:\psi^i\psi^j:(z), \tag{2}$$

where T^a_{ij} are generators of $\mathfrak{so}(N)$ in the fundamental representation. We will also employ the dimension -1 chiral operators,

$$\bar{J}^a(z) = \eta\partial\eta J^a(z), \tag{3}$$

with normal ordering being implicit. (We follow here the notation of [26] in which a related operator appears.) With this free field realization, one finds the OPEs

$$\begin{aligned}
J^a(z)J^b(w) &\sim \frac{\delta^{ab}}{(z-w)^2} + \frac{\mathrm{i}f^{ab}_c J^c(w)}{z-w} + O[(z-w)^0],\\
J^a(z)\bar{J}^b(w) &\sim \frac{\delta^{ab}\eta\partial\eta(w)}{(z-w)^2} + \frac{\mathrm{i}f^{ab}_c \bar{J}^c(w)}{z-w} + O[(z-w)^0],\\
\bar{J}^a(z)\bar{J}^b(w) &\sim \frac{(z-w)^2\delta^{ab}}{12}\eta\partial\eta\partial^2\eta\partial^3\eta(w) + \frac{\mathrm{i}f^{ab}_c(z-w)^3}{12}\eta\partial\eta\partial^2\eta\partial^3\eta J^c(w) + O[(z-w)^4].
\end{aligned} \tag{4}$$

J^a and $\bar{J}^a$ will enter the operators dual to positive and negative helicity gluons, respectively.

Since η has weight $h = -3/2$, it has four zero modes on the sphere. Correlation functions of J^a and $\bar{J}^a$ must have exactly 2 $\bar{J}^a$ insertions to soak up these zero modes [26]. The relevant correlation function $\langle \eta\partial\eta(z_1)\eta\partial\eta(z_2)\rangle$ is computed as follows. The globally holomorphic zero modes of η are

$$\eta = \eta_{3/2} + \eta_{1/2}z + \eta_{-1/2}z^2 + \eta_{-3/2}z^3. \tag{5}$$

These are the only modes contributing to the correlator, and lead to

$$\langle \eta\partial\eta(z_1)\eta\partial\eta(z_2)\rangle = z_{12}^4, \tag{6}$$

where we adopted the convention $\int \mathrm{d}\eta_n\eta_n = 1$ for the Grassmann measure.

The n-point nonvanishing $J^a, \bar{J}^a$ correlators are

$$\langle \bar{J}^{a_1}(z_1)\bar{J}^{a_2}(z_2)\prod_{j=3}^{n} J^{a_j}(z_j)\rangle = \mathrm{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n})\frac{z_{12}^4}{z_{12}z_{23}\cdots z_{n1}} + \cdots, \tag{7}$$

where $\cdots$ includes other color orderings and multitrace terms. At leading order in the large N limit only the first term contributes to the color-ordered correlator, which we denote with suppressed gauge indices,

$$\langle \bar{J}(z_1)\bar{J}(z_2)\prod_{j=3}^{n} J(z_j)\rangle = \frac{z_{12}^4}{z_{12}z_{23}\cdots z_{n1}}. \tag{8}$$

Liouville sector: We consider Liouville theory with central charge

$$c_L = 1 + 6Q^2 = 1 + 6(b + b^{-1})^2, \tag{9}$$

and coupling μ. This theory contains the primary operators

$$V_\alpha(z,\bar{z}) = \mathrm{e}^{2\alpha\phi(z,\bar{z})}, \tag{10}$$

where ϕ is the Liouville field. They are said to have "momentum" α and have conformal weight

$$\Delta(V_\alpha) = 2\alpha(Q-\alpha). \tag{11}$$

A brief review of the relevant aspects of Liouville is provided in the Supplemental Material.

We are interested in correlation functions of light operators in the classical limit of Liouville theory. These operators have momenta $\alpha = b\sigma$ scaling as b in the $b \to 0$ limit. Using the techniques of [31–33], we show in the Supplemental Material that these are proportional to n-point contact Witten diagrams in hyperbolic 3-space H^3,

$$\left\langle \prod_{j=1}^{n} V_{b\sigma_j}(z_j,\bar{z}_j)\right\rangle = \frac{\mathrm{e}^{-2\gamma_E+2/b^2}\lambda^{1/b^2}}{\pi b^3}\csc\left(\pi\left(b^{-2}-\frac{1}{2}\beta\right)\right)\lambda^{-1-\frac{\beta}{2}}\mathcal{C}_{2\sigma_1,\ldots,2\sigma_n}. \tag{12}$$

Here, γ_E is the Euler-Mascheroni constant, $\lambda = \pi\mu b^2$ is held fixed as we send $b \to 0$, and β is given by the sum over weights

$$\beta = 2\left(\sum_j \sigma_j - 2\right), \tag{13}$$

and the contact Witten diagrams are given by

$$\mathcal{C}_{2\sigma_1,\ldots,2\sigma_n}(z_i,\bar{z}_i) = \int_{H^3}\mathrm{D}^3x\prod_{j=1}^{n}G_{2\sigma_j}(z_j,\bar{z}_j;x), \tag{14}$$

with x denoting coordinates and D^3x the measure on the unit hyperboloid. The integrand is a product of scalar bulk-to-boundary propagators $G_{2\sigma_j}(z_j,\bar{z}_j;x)$ of weight $2\sigma_j$. (This relation between semiclassical Liouville and Witten diagrams appears to be new: related formulas appear in [31,32].) In the special case of $n=3$, we have the simple formula

$$\mathcal{C}_{2\sigma_1,2\sigma_2,2\sigma_3} = \frac{\pi}{2}\frac{\Gamma(\sigma_1+\sigma_2+\sigma_3-1)\Gamma(\sigma_1+\sigma_2-\sigma_3)\Gamma(\sigma_2+\sigma_3-\sigma_1)\Gamma(\sigma_3+\sigma_1-\sigma_2)}{\Gamma(2\sigma_1)\Gamma(2\sigma_2)\Gamma(2\sigma_3)(z_{12}\bar{z}_{12})^{\sigma_1+\sigma_2-\sigma_3}(z_{23}\bar{z}_{23})^{\sigma_2+\sigma_3-\sigma_1}(z_{31}\bar{z}_{31})^{\sigma_3+\sigma_1-\sigma_2}}. \tag{15}$$

Celestial amplitudes from dressed Liouville correlators—In this section, we explain the holographic dictionary from the 2D dressed Liouville theory to the 4D MHV amplitude. From this point on, we simply follow the prescription detailed in [20–22] for the construction of celestial amplitudes from leaf amplitudes. We sketch it here but refer the reader to those references for details.

Dressed Liouville → Euclidean leaves: The first step is to state the dictionary between bulk conformal primary gluons and boundary dressed Liouville operators. We posit

$$\mathcal{O}_\Delta^{+a,\varepsilon}(z,\bar{z}) = \mathrm{e}^{-\mathrm{i}\varepsilon\pi(\Delta-1)/2}\lim_{b\to 0}N_\Delta^{+}J^a(z)V_{b(\Delta-1)/2}(z,\bar{z}),$$
$$\mathcal{O}_\Delta^{-a,\varepsilon}(z,\bar{z}) = \mathrm{e}^{-\mathrm{i}\varepsilon\pi(\Delta+1)/2}\lim_{b\to 0}N_\Delta^{-}\bar{J}^a(z)V_{b(\Delta+1)/2}(z,\bar{z}), \tag{16}$$

where $\varepsilon = \pm 1$ labels whether the gluon is outgoing or ingoing and

$$N_\Delta^{-} = \lambda^{\Delta/2}\mathrm{e}^{\gamma_E}\sqrt{\mathrm{e}^{-2/b^2}\lambda^{-1/b^2}\pi b^3\sin(\pi/b^2)}\Gamma(\Delta+1)\mathrm{e}^{-\mathrm{i}\pi(\Delta+1)/2}$$
$$N_\Delta^{+} = \lambda^{(\Delta-1)/2}\Gamma(\Delta-1)\mathrm{e}^{-\mathrm{i}\pi(\Delta-1)/2}. \tag{17}$$

Of course there are other operators in the dressed Liouville theory that are not of this form. While they may have a bulk interpretation [34], we do not supply an interpretation of such herein. Bulk amplitudes of $\mathcal{O}_\Delta^{\pm a,\varepsilon}$ are then given by 2D CFT correlators of the operators on the right-hand side. Taking the $b \to 0$ limit one finds the color-ordered correlator [35]

$$\left\langle \mathcal{O}_{\Delta_1}^{-,\varepsilon_1}(z_1,\bar{z}_1)\mathcal{O}_{\Delta_2}^{-,\varepsilon_2}(z_2,\bar{z}_2)\prod_{j=3}^{n}\mathcal{O}_{\Delta_j}^{+,\varepsilon_j}(z_j,\bar{z}_j)\right\rangle$$
$$= \prod_j \mathrm{e}^{-\mathrm{i}\pi\varepsilon_j\bar{h}_j}\Gamma(2\bar{h}_j)\frac{z_{12}^4}{z_{12}\cdots z_{n1}}\mathcal{C}_{2\bar{h}_1,2\bar{h}_2,\ldots,2\bar{h}_n}. \tag{18}$$

As shown in [20], this is precisely the expression for the MHV *leaf* amplitude obtained by integrating the gluon interaction over a single spacelike (but asymptotically null) H^3 slice in Minkowski space.

We conclude that the dressed Liouville theory supplied with the dictionary (16) correctly generates the Minkowskian H^3 leaf amplitudes.

Euclidean leaves → Lorentzian leaves: The full Minkowskian celestial amplitudes are given by integrals of the leaf amplitudes over all the leaves of a hyperbolic foliation of Minkowski space. These necessarily include integrals over the 3D de Sitter leaves of the region spacelike separated from the origin as well as the H^3 leaves in the Milne region. Rather than compute 3D de Sitter leaf amplitudes, we choose the simpler route of analytically continuing from Minkowski to Klein space, which directly yields the Kleinian celestial amplitudes. The hyperbolic slices or leaves are then all Lorentzian $\mathrm{AdS}_3/\mathbb{Z}$ geometries and divide into two wedges containing points that are either timelike or spacelike separated from the origin.

Analytically continuing to Klein space $\mathbb{R}^{2,2}$ induces a corresponding continuation in the leaf amplitudes. In particular, the integrals over H^3 in the Witten diagram (14) and its past-pointing counterpart become an integral over $\mathrm{AdS}_3/\mathbb{Z}$ with an $i\epsilon$ prescription in the bulk-to-boundary propagators. This reproduces exactly the detailed formulas found in [20] for the n-point MHV *leaf* amplitudes. The leaf amplitudes now live on the boundary of Lorentzian $\mathrm{AdS}_3/\mathbb{Z}$, known as the celestial torus.

Lorentzian leaves → celestial amplitudes: Finally, to get from leaf to celestial amplitudes, the holographic dictionary dictates that we extract the limit of the leaf amplitudes when the net conformal weights of the external particles obey

$$\beta \equiv \sum_j (\Delta_j - 1) = 0. \tag{19}$$

This projection onto $\beta = 0$ is performed by multiplying the leaf amplitudes with $\delta(\beta)$. When combined with its image under exchanging the timelike and spacelike cycles on the celestial torus, and division by $8\pi^3$, one recovers exactly the n-gluon MHV celestial amplitudes $\mathcal{A}$ as an integral over Klein space [20],

$$\begin{aligned}
&\frac{\delta(\beta)}{8\pi^3}\left(\left\langle \mathcal{O}^{-,\varepsilon_1}_{\Delta_1}(z_1,\bar z_1)\mathcal{O}^{-,\varepsilon_2}_{\Delta_2}(z_2,\bar z_2)\prod_{j=3}^n \mathcal{O}^{+,\varepsilon_j}_{\Delta_j}(z_j,\bar z_j)\right\rangle + (\bar z_i \leftrightarrow -\bar z_i)\right)\\
&\quad= \frac{z_{12}^3}{z_{23}\cdots z_{n1}}\int_0^\infty \frac{d\tau\,\tau^{-1-\beta}}{(2\pi)^4}\left(\int_{\hat x^2=-1} \mathrm{D}^3\hat x \prod_{j=1}^n \frac{\Gamma(2\bar h_j)}{(\epsilon - i\varepsilon_j q(z_j,\bar z_j)\cdot \hat x)^{2\bar h_j}} + \int_{\hat x^2=1}\mathrm{D}^3\hat x\prod_{j=1}^n \frac{\Gamma(2\bar h_j)}{(\epsilon - i\varepsilon_j q(z_j,\bar z_j)\cdot \hat x)^{2\bar h_j}}\right)\\
&\quad= \frac{z_{12}^3}{z_{23}\cdots z_{n1}}\int \frac{d^4x}{(2\pi)^4}\prod_{j=1}^n \frac{\Gamma(2\bar h_j)}{(\epsilon - i\varepsilon_j q(z_j,\bar z_j)\cdot \hat x)^{2\bar h_j}} = \mathcal{A}(1^-_{\Delta_1}2^-_{\Delta_2}3^+_{\Delta_3}\cdots n^+_{\Delta_n}),
\end{aligned} \tag{20}$$

where $q^\mu(z,\bar z) = (1 - z\bar z, z + \bar z, 1 + z\bar z, z - \bar z)$ denotes the embedding of the celestial torus in Klein space.

Acknowledgments—We are grateful to Eduardo Casali, Scott Collier, Laurent Freidel, Noah Miller, Nathan Seiberg, and Tomasz Taylor for useful conversations. This work was supported by DOE Grant No. de-sc/0007870, NSF GRFP Grant No. DGE1745303, the Simons Collaboration on Celestial Holography, and the Gordon and Betty Moore Foundation and the John Templeton Foundation via the Black Hole Initiative.

[1] K. Costello and N. M. Paquette, Celestial holography meets twisted holography: 4d amplitudes from chiral correlators, J. High Energy Phys. 10 (2022) 193.
[2] W. Fan, A. Fotopoulos, S. Stieberger, T. R. Taylor, and B. Zhu, Elements of celestial conformal field theory, J. High Energy Phys. 08 (2022) 213.
[3] E. Casali, W. Melton, and A. Strominger, Celestial amplitudes as AdS-Witten diagrams, J. High Energy Phys. 11 (2022) 140.
[4] R. Gonzo, T. McLoughlin, and A. Puhm, Celestial holography on Kerr-Schild backgrounds, J. High Energy Phys. 10 (2022) 073.
[5] S. Stieberger, T. R. Taylor, and B. Zhu, Yang-Mills as a Liouville theory, Phys. Lett. B **846**, 138229 (2023).
[6] K. Costello, N. M. Paquette, and A. Sharma, Top-down holography in an asymptotically flat spacetime, Phys. Rev. Lett. **130**, 061602 (2023).
[7] S. Stieberger, T. R. Taylor, and B. Zhu, Celestial Liouville theory for Yang-Mills amplitudes, Phys. Lett. B **836**, 137588 (2023).
[8] W. Melton, S. A. Narayanan, and A. Strominger, Deforming soft algebras for gauge theory, J. High Energy Phys. 03 (2023) 233.
[9] R. Bittleston, S. Heuveline, and D. Skinner, The celestial chiral algebra of self-dual gravity on Eguchi-Hanson space, J. High Energy Phys. 09 (2023) 008.
[10] K. Costello, N. M. Paquette, and A. Sharma, Burns space and holography, J. High Energy Phys. 10 (2023) 174.
[11] T. Adamo, W. Bu, and B. Zhu, Infrared structures of scattering on self-dual radiative backgrounds, J. High Energy Phys. 06 (2024) 076.
[12] A. Ball, S. De, A. Yelleshpur Srikant, and A. Volovich, Scalar-graviton amplitudes and celestial holography, J. High Energy Phys. 02 (2024) 097.
[13] W. Melton, F. Niewinski, A. Strominger, and T. Wang, Hyperbolic vacua in Minkowski space, arXiv:2310.13663.
[14] A. Sharma, Ambidextrous light transforms for celestial amplitudes, J. High Energy Phys. 01 (2022) 031.

[15] C. Jorge-Diaz, S. Pasterski, and A. Sharma, Celestial amplitudes in an ambidextrous basis, J. High Energy Phys. 02 (2023) 155.

[16] W. Fan, A. Fotopoulos, S. Stieberger, T. R. Taylor, and B. Zhu, Conformal blocks from celestial gluon amplitudes, J. High Energy Phys. 05 (2021) 170.

[17] W. Fan, A. Fotopoulos, S. Stieberger, T. R. Taylor, and B. Zhu, Conformal blocks from celestial gluon amplitudes. Part II. Single-valued correlators, J. High Energy Phys. 11 (2021) 179.

[18] C.-M. Chang, W. Cui, W.-J. Ma, H. Shu, and H. Zou, Shadow celestial amplitudes, J. High Energy Phys. 02 (2023) 017.

[19] C.-M. Chang and W.-J. Ma, Missing corner in the sky: Massless three-point celestial amplitudes, J. High Energy Phys. 04 (2023) 051.

[20] W. Melton, A. Sharma, and A. Strominger, Celestial leaf amplitudes, J. High Energy Phys. 07 (2024) 132.

[21] W. Melton, A. Sharma, and A. Strominger, Soft algebras for leaf amplitudes, J. High Energy Phys. 07 (2024) 070.

[22] W. Melton, A. Sharma, and A. Strominger, Conformal correlators on the Lorentzian torus, Phys. Rev. D **109**, L101701 (2024).

[23] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.133.091603 for a brief review of semiclassical correlators of Liouville theory.

[24] F. Cachazo and A. Strominger, Evidence for a new soft graviton theorem, arXiv:1404.4091.

[25] In AdS_3 holography, finite c corrections correspond to loop corrections in the bulk theory. It is tempting to speculate that finite-c corrections may play a similar role here, but it is not obvious why the 2D theory given here should be the full story beyond the MHV sector. Non-MHV amplitudes arise even at tree level in the bulk, so to get all the amplitudes will require some modification of the 2D CFT.

[26] W. Bu and E. Casali, The 4d/2d correspondence in twistor space and holomorphic Wilson lines, J. High Energy Phys. 11 (2022) 076.

[27] V. P. Nair, A current algebra for some gauge theory amplitudes, Phys. Lett. B **214**, 215 (1988).

[28] T. R. Taylor and B. Zhu, Celestial supersymmetry, J. High Energy Phys. 06 (2023) 210.

[29] G. Giribet, Remarks on celestial amplitudes and Liouville theory, arXiv:2403.03374.

[30] L. F. Alday, D. Gaiotto, and Y. Tachikawa, Liouville correlation functions from four-dimensional gauge theories, Lett. Math. Phys. **91**, 167 (2010).

[31] D. Harlow, J. Maltz, and E. Witten, Analytic continuation of Liouville theory, J. High Energy Phys. 12 (2011) 071.

[32] N. Seiberg, Notes on quantum Liouville theory and quantum gravity, Prog. Theor. Phys. Suppl. **102**, 319 (1990).

[33] L. Freidel, K. Krasnov, and E. R. Livine, Holomorphic factorization for a quantum tetrahedron, Commun. Math. Phys. **297**, 45 (2010).

[34] W. Melton and S. A. Narayanan, Celestial gluon amplitudes from the outside in, J. High Energy Phys. 05 (2024) 211.

[35] Here, we take $\lim_{b\to 0}[\sin(\pi/b^2)/\sin\pi(1/b^2-\beta/2)] = e^{i\pi\beta/2}$, which follows if we take the limit with a small fixed phase for b. We note also that the celestial amplitudes are projected onto $\beta = 0$ where this ratio is unity even before taking the limit.