Emergence of String Monodromy in Effective Field Theory

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We demonstrate that when the field theory Kleiss-Kuijf and Bern-Carrasco-Johansson relations are imposed on one color ordering of general local bi-adjoint scalar effective field theories, the string monodromy relations must be obeyed by the other color ordering. This is a surprising example of an open string world-sheet property appearing in a purely field theoretic context. As part of the derivation, we show that nonlinear relations among the four-point Wilson coefficients arise from imposing linear symmetries on the six-point amplitude via factorization.

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Introduction—In effective field theory (EFT), higher-derivative operators may involve group-theory structures that are different from those of the leading-order theory. This can affect linear relations obeyed by the EFT amplitudes. For example, the single-trace color-ordered gluon tree amplitudes of Yang-Mills theory obey a set of linear relations known as the Kleiss-Kuijf (KK) and Bern-Carrasco-Johansson (BCJ) relations. The KK relations include the photon decoupling identity and trace reversal identities such as $A_n[123\ldots n] = (-1)^n A_n[1n\ldots 32]$ [1,2]. The BCJ relations are related to color-kinematic duality [3]. In a YM EFT, the trF^3 operator has the same color structure as the interaction terms of trF^2, so the KK and BCJ relations (henceforth "KKBCJ relations") are unaffected. However, as shown in Ref. [4], the KKBCJ relations no longer hold when trF^4 is included. *Imposing* the KKBCJ relations on the YM EFT amplitudes restricts the Wilson coefficients of the higher-derivative operators; e.g., the couplings of trF^4 must vanish [4].

In this Letter, we study the result of imposing KKBCJ relations on the tree amplitudes of a generic massless bi-adjoint scalar (BAS) theory with higher-derivative terms ("BAS EFT") in d dimensions with $d > 3$. The *cubic* BAS model

$$\mathcal{L}_{\rm BAS} = -\frac{1}{2}(\partial\phi^{aa'})^2 + \frac{1}{6} g f^{abc}\tilde{f}^{a'b'c'}\phi^{aa'}\phi^{bb'}\phi^{cc'}, \quad (1)$$

plays a key role for the field theory double copy [6], as discussed in Refs. [7–9]. It has doubly color-ordered tree amplitudes $m_n^{(0)}[\alpha|\beta]$ that satisfy the KKBCJ relations for both color orderings α and β. When generic local higher-derivative terms are added to the cubic BAS model, there are in general no linear relationships among the amplitudes beyond the cyclicity inferred from the color traces. The absence of linear relations can be stated as the matrix $\mathbf{m}_n$ of BAS EFT tree amplitudes $m_n[\alpha|\beta]$ having full rank. When we *impose* KKBCJ relations on one color ordering (say β), the reduced rank of the matrix of tree amplitudes implies that the amplitudes must also obey a set of linear relations associated with the *other* color ordering (i.e., α). With KKBCJ imposed at four point only, the *second* set of linear relations depends on the Wilson coefficients at essentially each derivative order of the BAS EFT and does not reveal any particular structure. However, when KKBCJ is also imposed at six point, we find that the factorization channels involving the four-point vertices place additional nonlinear constraints among the four-point couplings. As a result, the second set of linear relations reduce to a much simpler form and, unlike the four-point amplitude itself, they depend only on the lowest-dimension EFT coupling. Specifically, we show for the four-, five-, and six-point BAS EFT amplitudes that the second set of linear relations are identical to the low-energy expansion of the string monodromy relations. The dimensionful coupling of the lowest-dimension operator in the BAS EFT sets the scale associated with α'.

In string theory, the monodromy relations are derived from the world-sheet description, and it is therefore very surprising that they appear out of a purely field theoretic analysis of an EFT. Note that the surprise is not that the second set of linear relations *include* the monodromy relations—that had to be the case because of the existence of Z-theory amplitudes (see the factorization, exponentiation, and resummation section).

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The surprise is that it is *precisely* the monodromy relations that arise from the field theory analysis and nothing else.

Review: KKBCJ and monodromy—KKBCJ relations: Consider the single-trace color-ordered amplitudes $A_n[\alpha]$, e.g., the tree amplitudes of YM theory. The KKBCJ relations are a set of

$$C_n \equiv (n-1)! - (n-3)! \tag{2}$$

independent linear relations among n-point amplitudes with different color orderings. They can be written

$$\sum_\alpha A_n[\alpha] N_n^I[\alpha] = 0 \tag{3}$$

with the sum over the $(n-1)!$ cyclic color orderings α. The $(n-1)!$-component *KKBCJ vectors* $N_n^I[\alpha]$, labeled by $I = 1, 2, \ldots, C_n$, each give one of the C_n KKBCJ relations.

At $n = 4$, the $C_4 = 5$ KKBCJ relations consist of three reflection identities, the photon decoupling relation, and a BCJ relation. We choose the six cyclic color orderings to be $\{1234, 1243, 1324, 1342, 1423, 1432\}$ and write in this basis the five KKBCJ vectors as

$$\begin{aligned} &N_4^1 = (1,0,0,0,0,-1), \qquad N_4^4 = (1,1,1,0,0,0),\\ &N_4^2 = (0,1,0,-1,0,0), \qquad N_4^5 = (u,-t,0,0,0,0),\\ &N_4^3 = (0,0,1,0,-1,0), \end{aligned} \tag{4}$$

with Mandelstams $s = s_{12}$, $t = s_{13}$, $u = s_{14}$, such that $s + t + u = 0$ and $s_{ij} = (p_i + p_j)^2$. Higher-point KKBCJ relations can be found in Ref. [10].

String monodromy relations: Type I open string tree-level scattering processes are disk amplitudes with n vertex operators inserted on the boundary of the disk. Amplitudes with distinct external ordering of the vertex operators correspond to different choices of contours in the integrals over the insertion points. Contour deformations relate the different color orderings and the resulting linear relations are the *string monodromy relations* [11–15]. With massless external states, the four-point monodromy relations take the form

$$A_4[1324] + e^{i\pi\alpha' u} A_4[1234] + e^{-i\pi\alpha' t} A_4[1342] = 0. \tag{5}$$

Exponentials of Mandelstams appear because the integrand is not single valued on the string moduli space. Crucially, the monodromy relations are a consequence of the string worldsheet. Thus, in a general EFT, there is no reason for the tree amplitudes to obey monodromy relations.

Splitting (5) into its real and imaginary parts, the monodromy relations can be combined with the trace reversal identities to give five linear relations among the open string amplitudes. They can be written in the form of Eq. (3) in which the vectors N_4^1, N_4^2, N_4^3 are the same as in Eq. (4), but N_4^4 and N_4^5 are replaced by

$$\begin{aligned} &N_4^{4,\mathrm{str}} = \left(1, 1, -\frac{\sin(\pi\alpha' t) + \sin(\pi\alpha' u)}{\sin(\pi\alpha' s)}, 0, 0, 0\right),\\ &N_4^{5,\mathrm{str}} = \left(\sin(\pi\alpha' u), -\sin(\pi\alpha' t), 0, 0, 0, 0\right). \end{aligned} \tag{6}$$

When $\alpha' = 0$, the string monodromy relations (6) reduce to the field theory KKBCJ relations (4).

Setup—We consider a Lorentz-invariant local d-dimensional model with a massless bi-adjoint scalar field $\phi^{aa'}$. It carries adjoint indices under two non-Abelian global "color"-symmetry groups, G and G'. We assume the model to have a canonical kinetic term and the interactions of interest are single trace in each color structure. The tree amplitudes $m_n[\alpha|\beta]$ are doubly color ordered, with α (β) denoting the single-trace ordering of n generators associated with G (G').

We impose the KKBCJ relations on the second color ordering

$$\sum_\beta m_n[\alpha|\beta] N_n^I[\beta] = 0, \tag{7}$$

with I labeling the C_n KKBCJ vectors and the sum running over the $(n-1)!$ cyclically independent color orderings β.

The KKBCJ condition in Eq. (7) is the statement that the $(n-1)! \times (n-1)!$ matrix $\mathbf{m}_n$ of tree amplitudes $m_n[\alpha|\beta]$ has C_n null vectors N_n^I under right multiplication. (We use "null vector" here as a shorthand for a vector in the null space of the matrix.) It follows that $\mathbf{m}_n$ must have rank $(n-3)!$ and therefore $\mathbf{m}_n$ must also have C_n null vectors K_n^I under left multiplication, i.e.,

$$\sum_\alpha K_n^I[\alpha] m_n[\alpha|\beta] = 0. \tag{8}$$

A priori, it is not clear what form these vectors should take or how they depend on the Wilson coefficients. We are going to show that the null vectors K_4^I are highly restricted and match the α' expansion of $N_4^{I,\mathrm{str}}$ given in Eq. (6).

Imposing KKBCJ—The KKBCJ relations (7) impose constraints on the Wilson coefficients of the BAS EFT. We solve these order by order in the derivative expansion for three-, four-, five-, and six-point amplitudes.

Three point: A bi-adjoint scalar model can have two possible cubic self-interactions: the three pairs of adjoint indices can either be contracted with the antisymmetric structure constants f^{abc} or with the fully symmetric tensors $d^{abc} = \mathrm{Tr}[T^a\{T^b, T^c\}]$. The reversal symmetry constraint $m_3[123|123] = -m_3[123|132]$ rules out the d^{abc} contraction. In the absence of other interactions, the result is the *cubic BAS model* in Eq. (1).

Now add to the cubic BAS model all possible local single-trace (in G and G') higher-derivative interactions, schematically $\mathrm{tr}\,\partial^{2k}\phi^n$. On-shell, the n-field Lagrangian operators of this form are one-to-one with Mandelstam polynomial terms in the n-point matrix elements.

Four point: Using cyclicity and momentum relabeling, the $3! \times 3! = 36$ possible color-ordered amplitudes $m_4[\alpha|\beta]$ can be written in terms of three functions [9]:

$$f_1(s,t) = m_4[1234|1234], \qquad f_2(s,t) = m_4[1234|1243],$$
$$f_6(s,t) = m_4[1234|1432]. \tag{9}$$

We now impose the KKBCJ relations from Eqs. (4) and (7). N_4^1 gives $f_6(s,t) = f_1(s,t)$ and then N_4^2 and N_4^3 are automatically satisfied. The N_4^4 and N_4^5 constraints are solved by

$$f_1(s,t) = \frac{t}{u} f_2(s,t) \quad \text{and} \quad u f_2(u,s) = t f_2(t,s). \tag{10}$$

These relations ensure that f_1 is cyclic, $f_1(u,t) = f_1(s,t)$. Setting the cubic coupling to $g = 1$ without loss of generality, we write the ansatz

$$f_2(s,t) = -\frac{1}{s} + \sum_{k=0}^{N} \sum_{r=0}^{k} a_{k,r} s^r t^{k-r}, \tag{11}$$

where terms up to $2N$ derivatives are included. Solving the second condition of Eq. (10) order by order in the momentum expansion, we find

$$\begin{aligned}
&a_{0,0} = 0, \qquad a_{1,1} = a_{1,0}, \\
&a_{2,1} = a_{2,0}, \qquad a_{2,2} = 0, \\
&a_{3,2} = 2a_{3,1} - 2a_{3,0}, \qquad a_{3,3} = a_{3,1} - a_{3,0}, \\
&a_{4,2} = 2a_{4,1} - 2a_{4,0}, \qquad a_{4,3} = a_{4,1} - a_{4,0}, \\
&a_{4,4} = 0, \qquad a_{5,3} = 5a_{5,0} - 5a_{5,1} + 3a_{5,2}, \\
&a_{5,4} = 6a_{5,0} - 6a_{5,1} + 3a_{5,2}, \\
&a_{5,5} = 2a_{5,0} - 2a_{5,1} + a_{5,2},
\end{aligned} \tag{12}$$

and it is straightforward to extend this to higher orders in the Mandelstams. There is no mixing between orders since the KKBCJ relations are linear equations. The construction ensures that the f_1 given by Eq. (10) does not have unphysical poles.

The five left-multiplication null vectors K_4^I of Eq. (8) consist of the three reversal-symmetry null vectors $K_4^I = N_4^I$, $I = 1, 2, 3$, and two other null vectors

$$\begin{aligned}
K_4^4 &= \left\{1, 1, -\frac{f_2(s,u)}{f_2(u,s)} - \frac{f_2(s,t)}{f_2(t,s)}, 0, 0, 0\right\}, \\
K_4^5 &= \left\{1, -\frac{f_2(u,t)}{f_2(t,u)}, 0, 0, 0, 0\right\},
\end{aligned} \tag{13}$$

that are generalizations of the photon decoupling identity and of the BCJ relations (see Ref. [9]). In general, the null vectors in Eq. (13) depend on most of the free $a_{k,r}$'s in f_2. This changes when constraints from the six-point analysis are implemented.

Five point: The $4! \times 4!$ amplitudes of $\mathbf{m}_5$ can be parametrized in terms of eight basis amplitudes $g_1, \ldots, g_8$ defined in Eq. (6.1) of Ref. [9]. For each g_i, we construct a consistent factorization to three- and four-point amplitudes and then include all possible polynomial terms in the Mandelstam variables to parametrize the possible local contact terms. Their coefficients are constrained by imposing the $C_5 = 22$ five-point KKBCJ relations, which we have solved to $O(s^{17})$. As an example, relevant in the following, the result for $g_4 \equiv m_5[12\,345|12\,543]$ is

$$\begin{aligned}
g_4[12\,345] = {}& \frac{1}{s_{12}s_{34}} + \frac{1}{s_{12}s_{45}} + a_{1,0}\left(-\frac{s_{35}}{s_{12}} + \frac{s_{15}}{s_{34}} + \frac{s_{23}}{s_{45}}\right) \\
&+ a_{2,0}\left(-\frac{s_{35}^2}{s_{12}} + \frac{s_{15}s_{25}}{s_{34}} + \frac{s_{13}s_{23}}{s_{45}} + 2s_{35}\right) + \ldots .
\end{aligned} \tag{14}$$

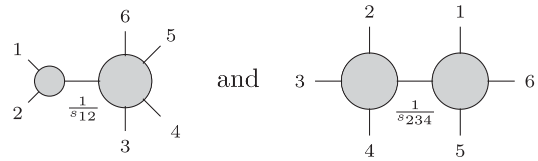
At $O(s^2)$, the expressions are longer, but all terms in the g_i are fixed at that order in terms of the four-point coefficients $a_{3,0}$ and $a_{3,1}$. At $O(s^3)$, we find that all local 5-point coefficients, except two, are fixed in terms of $a_{4,0}$ and $a_{4,1}$. At $O(s^4)$ there are five free parameters, one of which parametrizes a local term that violates reversal symmetry on the first color ordering, i.e., $m_5[\alpha^T|\beta] \neq -m_5[\alpha|\beta]$, where α^T is the reverse of the color order α [e.g., $(12\,345)^T = (54\,321)$]. Importantly, we find no constraints on any 4-point Wilson coefficients $a_{k,r}$ from the five-point KKBCJ relations.

Six point:The $5! \times 5!$ amplitudes of the matrix $\mathbf{m}_6$ are written in terms of 24 basis amplitudes that are independent under cyclicity and momentum relabeling. We construct the most general ansatz for the basis amplitudes consistent with all factorization channels and with arbitrary coefficients for the local six-point contact terms. We impose the $C_6 = 114$ 6-point KKBCJ relations and solve them systematically order by order in the Mandelstams. Remarkably, this constrains the four-point coefficients $a_{k,r}$. For example, we find that

$$a_{3,0} = -\frac{2}{5} a_{1,0}^2, \qquad a_{3,1} = -\frac{11}{10} a_{1,0}^2. \tag{15}$$

Together with the 4-point constraints in Eq. (12), this completely fixes the $O(s^3)$ terms in the f_2 ansatz of Eq. (11) in terms of just $a_{1,0}$.

Since the KKBCJ relations are linear, any nonlinear relations, such as in Eq. (15), that mix orders in the four-point expansion have to arise from combinations of different factorization channels in the six-point amplitudes. With their explicit Mandelstam factors, only the BCJ relations allow such mixing of diagrams. Thus, to illuminate and extend the result in Eq. (15), we consider special kinematic limits of the BCJ relations to isolate pole contributions from diagrams such as



(16)

Consider the $O(s)$ terms from these diagrams. In the first diagram, this comes from the $O(s^2)$ terms in the five-point amplitude that are completely determined by $a_{3,0}$ and $a_{3,1}$, as noted below Eq. (14). In contrast, the product of four-point amplitudes in the second diagram has contributions with coefficients $a_{1,0}^2$ as well as $a_{3,0}$ and $a_{3,1}$. Thus, a BCJ relation involving two such diagrams can result in a relation such as Eq. (15).

We can make this argument precise by isolating the pole contributions in the BCJ relations via carefully chosen kinematic limits. Consider the following two BCJ relations:

$$\begin{aligned} 0 = {} & s_{34}s_{234}m_6[\mathbb{1}|134265] + s_{46}(s_{12}+s_{25})m_6[\mathbb{1}|136425] + s_{46}s_{12}m_6[\mathbb{1}|136452] + (s_{46}+s_{45})s_{12}m_6[\mathbb{1}|136542] \\ & + (s_{14}+s_{45})(s_{234}+s_{26})m_6[\mathbb{1}|136245] + s_{14}(s_{234}+s_{26})m_6[\mathbb{1}|136254] + s_{14}(s_{234}+s_{26}+s_{25})m_6[\mathbb{1}|136524] \end{aligned} \tag{17}$$

$$\Rightarrow 0 = \left[s_{34}f_2(s_{34},s_{24})f_2(s_{56},s_{15}) - s_{45}g_4[P_{12}3456]\right]\Big|_{s_{14},s_{26},s_{46},s_{234},s_{12}=0}, \tag{18}$$

$$\begin{aligned} 0 = {} & s_{23}s_{234}m_6[\mathbb{1}|132456] + s_{25}(s_{14}+s_{46})m_6[\mathbb{1}|135246] + s_{25}s_{14}m_6[\mathbb{1}|135264] + (s_{25}+s_{26})s_{14}m_6[\mathbb{1}|135624] \\ & + (s_{12}+s_{26})(s_{234}+s_{45})m_6[\mathbb{1}|135426] + s_{12}(s_{234}+s_{45})m_6[\mathbb{1}|135462] + s_{12}(s_{234}+s_{45}+s_{46})m_6[\mathbb{1}|135642] \end{aligned} \tag{19}$$

$$\Rightarrow 0 = \left[s_{23}\Big(f_2(s_{23},s_{24})f_1(s_{16},s_{15})\Big) - s_{26}g_3[6123P_{45}] + f_1(s_{123},s_{36}) - s_{46}g_3[P_{12}3456]\right]\Bigg|_{\substack{s_{25},s_{14},s_{234},\\ s_{12},s_{45}}=0} \tag{20}$$

To obtain these equations, the BCJ relations in Eq. (4.27) of Ref. [3] are imposed on the β color ordering of the BAS EFT tree amplitudes $m_6[\alpha|\beta]$ for any choice of the color ordering α. Equation (17) is obtained by choosing $\alpha = 154\,263$ and then relabeling $[154\,263] \to [123\,456]$ and Eq. (19) arises from $\alpha = 142\,536$ and then relabeling $[142\,536] \to [123\,456]$. The notation $\mathbb{1}$ is shorthand for the color-ordering 123 456. To get Eq. (18), take the limit s_{14}, s_{26}, $s_{46} \to 0$ of Eq. (17) (the amplitudes have no poles in those variables) and then pickup the residues from poles at $s_{234} = 0$ and $s_{12} = 0$. Equation (18) precisely realizes the relation between the two diagrams shown in Eq. (16). A similar kinematic limit of Eq. (19) gives Eq. (20).

Expanding Eqs. (18) and (20) in Mandelstams, we find to $O(s^3)$ that

$$\begin{aligned} 0 &= (a_{1,0}^2 - 3a_{3,0} + 2a_{3,1})s_{16}s_{23}s_{34}, \\ 0 &= (-a_{1,0}^2 - 8a_{3,0} + 2a_{3,1})s_{23}s_{34}(s_{16}+s_{56}), \end{aligned} \tag{21}$$

from which Eq. (15) follows. For the next two powers in the Mandelstams, we find that $a_{4,0}$ remains unfixed but

$$\begin{aligned} a_{4,1} &= -a_{1,0}a_{2,0} + 2a_{4,0}, & a_{5,0} &= \frac{8}{35}a_{1,0}^3, \\ a_{5,1} &= \frac{34}{35}a_{1,0}^3 - \frac{1}{2}a_{2,0}^2, & a_{5,2} &= \frac{27}{14}a_{1,0}^3 - a_{2,0}^2. \end{aligned} \tag{22}$$

Together with the four-point results in Eq. (12), we have found, up to $O(s^{18})$, that Eqs. (18) and (20) fix all four-point coefficients except $a_{1,0}$ and $a_{2k,0}$. Moreover, all 5-point local coefficients up to order $O(s^6)$ are fixed too by the six-point KKBCJ constraints; we find, in particular, that operators violating reversal symmetry are eliminated.

Unexpected monodromies—With the constraints from the four-, five-, and six-point analysis—i.e., Eq. (15), Eq. (22), and those at higher orders—imposed on the Wilson coefficients $a_{k,q}$ in f_2, we evaluate the left-multiplication null vectors K_4^I in Eq. (13). We find that the $a_{2k,0}$'s drop out from the ratios of f_2 so that the K_4^I only depend on a single parameter, namely, $a_{1,0}$. Order by order in the Mandelstam expansion, we find that the polynomial dependence on the Mandelstam variables in K_4^I is identical to that of the α' expansion of the monodromy vectors $N_4^{I,\mathrm{str}}$ from Eq. (6). Moreover, we find that the K_4^I's are in fact equal to the α' expansion of the $N_4^{I,\mathrm{str}}$'s when we set

$$a_{1,0} = -\frac{\pi^2}{6}\alpha'^2, \tag{23}$$

as checked up to $O(s^{18})$.

Note that Eq. (23) is merely a choice of scale of α' in terms of the Wilson coefficient of the lowest-order EFT operator. Thus, the BAS EFT amplitude is required to obey the string monodromy relations. This also holds for the five- and six-point amplitudes as checked to $O(s^6)$ and $O(s^4)$, respectively. Note that the K_n^I vectors do not match the monodromies without constraints on the 4-point coefficients $a_{k,q}$ from the six-point analysis.

Factorization, exponentiation, and resummation—We can understand why the dependence on the $a_{2k,0}$'s drop out from the K_4^I's by examining the structure of f_2 more closely. It turns out f_2 factorizes as

$$f_2 = f_2^{(0)} U, \qquad f_2^{(0)} = f_2|_{a_{2k,0}=0}, \tag{24}$$

where all the $a_{2k,0}$ dependence is contained in the function U which is fully symmetric in s, t, u. Hence ratios of the f_2's in the K_4^I null vectors of Eq. (13) are independent of U and hence of the $a_{2k,0}$'s.

Moreover, from the low-energy expansion (to 18th order in the Mandelstams), we find that U can be resummed into the form $U = e^V$ with

$$V(s,t,u) = \sum_{k=1}^{\infty} \frac{a_{2k,0}}{2k+1} (s^{2k+1} + t^{2k+1} + u^{2k+1}). \tag{25}$$

Since there can be more than one symmetric polynomial in s, t, u starting at order 6, it is highly nontrivial that U takes this form, and it relies heavily on the constraints from the six-point analysis.

Z theory: Because the BAS EFT tree amplitudes obey KKBCJ relations on the 2nd color ordering, its amplitudes can be left-sector input in double-copy relations with the field theory KLT kernel Ref. [9]. For example, it can be double copied with YM theory to give YM + higher derivative (h.d.) amplitudes:

$$A_n^{\text{YM+h.d.}}[\alpha] = \sum_{\beta,\gamma} m_n[\alpha|\beta] S_n^{\text{FT}}[\beta|\gamma] A_n^{\text{YM}}[\gamma], \tag{26}$$

where the sum is over a choice two set of $(n-3)!$ color-orderings β and γ, and S_n^{FT} denotes the standard field theory KLT kernel. The property that $m_4[\alpha|\beta]$ obeys the string monodromy relations on the first color-ordering α, is inherited by the YM+h.d. tree amplitudes constructed by Eq. (26). Such amplitudes will include the open string tree amplitudes.

Indeed, the open string tree amplitudes are known to be constructible by a double-copy such as in Eq. (26) but with $m_n[\alpha|\beta]$ replaced by the Z-theory amplitudes $m_n^Z[\alpha|\beta]$. The Z-theory amplitude f_1 is the beta-function and f_2, found from the BCJ relation, is then

$$f_2^Z(s,t) = -\frac{1}{s} \frac{\Gamma(1+\alpha' s)\Gamma(1-\alpha'(s+t))}{\Gamma(1-\alpha' t)}. \tag{27}$$

The Z-theory amplitudes arise from period integrals in the computation of the disk amplitudes, see Ref. [5], and they are known to obey string monodromy on the first color-ordering and field-theory KKBCJ relations on the second one. Thus, Z theory is a special case of our BAS EFT with $a_{1,0}$ identified with α' as in Eq. (23) and

$$a_{2k,0} = -(\alpha')^{2k+1} \zeta(2k+1). \tag{28}$$

Note that the entire dependence on the odd-ζ's is contained in the symmetric function U. [The factorized form (24) exponentiation (25) with the coefficients of Eq. (28) was be found in Ref. [16].] For the choice in Eq. (28), the symmetric function V in Eq. (25) resums to a compact expression with logarithms of Γ functions. Because Eq. (24) gives $f_2^Z = f_2^{(0)} e^V$, this leads us to propose a re-summed form of $f_2^{(0)}$, namely,

$$f_2^{(0)} = -\frac{1}{s} \sqrt{\frac{\pi s(s+t)\alpha' \sin(\pi\alpha' t)}{t \sin(\pi\alpha' s) \sin[\pi\alpha'(s+t)]}}. \tag{29}$$

One can directly verify that this gives amplitudes that solve the string monodromy relations on the first color-ordering and KKBCJ on the second.

Discussion—We assumed in our analysis that the cubic coupling and $a_{1,0}$ are nonzero in the BAS EFT model. If we instead set $a_{1,0} = 0$, the left-multiplication null vectors become the field theory KKBCJ relations of the string monodromy relations, and it follows from Eq. (24) that the only solution for f_2 is then $-1/s$ times the symmetric function U. Choosing $a_{2k,0}$ to be twice the value for Z theory given in Eq. (28) these amplitudes are then the "J integrals," or period integrals, of the closed string in Ref. [17].

The results in this Letter are strictly tree level. The proposed monodromy relations for loop integrands studied in Refs. [18–25] may shed light on any potential generalization to loops.

The string monodromy relations guarantee that the KLT double copy of two open string amplitudes is independent of the choice of $(n-3)!$ color orderings in the double-copy sum; see Refs. [26] and [9]. The results presented in this Letter have significant consequences for generalizations of the double copy. In Ref. [27], we use the results of this Letter to argue that the most general KLT-like double copy *requires* the "input" tree amplitudes (e.g., Yang-Mills with higher-derivative corrections) to satisfy either the string monodromy relations or the KKBCJ relations; there are no other options.

We have shown how linear symmetry constraints at six point can result in nonlinear constraints among four-point EFT couplings. This new observation may have an impact on the EFT S-matrix bootstrap which uses unitarity and analyticity of $2 \to 2$ scattering amplitudes to derive upper and lower bounds on EFT couplings. The resulting allowed regions are convex because the addition of four-point amplitudes is compatible with the linearized unitarity constraints. In this Letter, we have demonstrated that the sum of two four-point amplitudes may not be compatible with linear symmetries at higher point due to factorization. This can significantly impact the resulting allowed regions in S-matrix bootstraps for models with additional symmetries, such as supersymmetry. These ideas will be explored in future work.

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