Gapped Boundaries of Fermionic Topological Orders and Higher Central Charges

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The existence of gapped boundaries of bosonic topological orders can be tested in terms of the vanishing of higher central charges, which can be easily computed in terms of the modular data. For fermionic topological orders, even the chiral central charge admits no simple expression in terms of the modular data. Using the congruence property of representations formed by the modular data, we develop a method that tests whether the higher central charges of a fermionic topological order, including the chiral central charge, vanish. The test can be carried out entirely in terms of the modular data of the super-modular tensor category describing the fermionic topological order, and does not require explicit computation of modular extensions. This leads to a stringent set of easily computable necessary conditions for a fermionic topological order to admit a gapped boundary. We apply our test to known examples of fermionic topological orders to determine which of them potentially admit a gapped boundary.

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At its most basic level, bulk-boundary correspondence for 2D topological orders controls the gappability of its boundary. The bulk hosts anyonic excitations whose braiding and fusion properties are described mathematically by modular tensor categories (MTCs) and their S and T matrices (collectively called the "modular data") [1–4], and only some MTCs are consistent with a gapped boundary.

For bosonic topological orders, gapped boundaries are now well understood. The boundary is gappable if and only if the bulk hosts a set of anyons that can simultaneously be *condensed* to yield the trivial vacuum—mathematically, these anyons form a *Lagrangian algebra* [5–9]. In practice, it is advantageous to compute simple quantities that obstruct the existence of a gapped boundary. The best known is the *chiral central charge* c (which determines the thermal Hall conductance), whose value modulo 8 is computed from the bulk data using the Gauss-Milgram formula

$$e^{2\pi i c/8} = \frac{1}{D}\sum_a d_a^2 \theta_a. \tag{1}$$

Here, the index a runs over anyon types, and d_a and θ_a are respectively quantum dimensions and topological spins of the anyons, and D is the total quantum dimension of the MTC. The boundary cannot be gapped unless c vanishes.

There are additional obstructions called *higher central charges* that are also easily computed in terms of the bulk data as [10–12]

$$\xi_n = \frac{\sum_a d_a^2 \theta_a^n}{|\sum_a d_a^2 \theta_a^n|}. \tag{2}$$

For the topological order to admit a gapped boundary, we need $\xi_n = 1$ for each n coprime to N_{FS} (called the "Frobenius-Schur exponent"), which is defined as the order of the T matrix [13]. We will say that a higher central charge ξ_n *vanishes* if $\xi_n = 1$. While the vanishing of higher central charges is only a necessary condition, it is a very stringent one, ruling out many topological orders (even with $c = 0$ mod 8) from admitting a gapped boundary in practice.

For fermionic topological orders, which contain a fundamental fermion as an excitation such that the bulk data is described by a super-modular tensor category (super-MTC) [16,17], no analogous efficient method to determine the gappability of its boundary had been known. Even the Gauss-Milgram formula has no straightforward fermionic analog, because the S matrix of a super-MTC is degenerate. The chiral central charge of a super-MTC is well-defined modulo $\frac{1}{2}$ in principle, but it is defined by reference to *modular extensions*: regular MTCs that contain the super-MTC as a subcategory [16,18,19], which may physically be thought of as the bosonized theories [20–22]. For a given unitary super-MTC $\mathcal{B}$, there are 16 distinct unitary modular extensions [19,28]. Each modular extension has well-defined c mod 8 via Eq. (1), and the 16 modular extensions have equivalent c mod $\frac{1}{2}$, which we define as the chiral central charge of $\mathcal{B}$. The computation of modular extensions, however, is highly nontrivial in general (though see Ref. [29] for explicit computations in low rank).

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The first result of the present Letter is a method to determine whether $c = 0 \bmod \frac{1}{2}$ intrinsically in terms of the super-MTC data, without explicit computation of modular extensions. This makes our test completely tractable and algorithmic.

As in the bosonic case, even if a super-MTC has $c = 0 \bmod \frac{1}{2}$, it is not necessarily compatible with a gapped boundary. By analogy to the bosonic case, we say that a fermionic topological order admits a gapped boundary if it admits a condensation to the trivial fermionic topological order [30]. Mathematically, this means the bulk super-MTC has to belong to the trivial *super-Witt class* [10,23,31,32]. Concretely, this means a given super-MTC $\mathcal{B}$ is compatible with a gapped boundary if and only if it has a modular extension $\breve{\mathcal{B}}$ that admits a gapped boundary in the bosonic sense. Such a $\breve{\mathcal{B}}$ needs $c = 0 \bmod 8$, which in turn requires that $\mathcal{B}$ has $c = 0 \bmod \frac{1}{2}$. However, there are additional necessary conditions coming from the higher central charges of $\breve{\mathcal{B}}$.

The second result of this Letter expresses the condition that all higher central charges of $\breve{\mathcal{B}}$ vanish purely in terms of the modular data of $\mathcal{B}$. This leads to a stringent set of necessary conditions for $\mathcal{B}$ to admit a gapped boundary.

Main result—We state our main result in the form of two theorems, which together provide a stringent set of easily computable necessary conditions for a given super-MTC to admit a gapped boundary.

First, note that the modular data of a super-MTC always admit the following tensor decomposition [17]:

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \hat{S}, \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \hat{T}. \quad (3)$$

While S is degenerate and there is no canonical choice of $\hat{T}$, $\hat{S}$ is unitary and $\hat{T}^2$ is well-defined. We will refer to $(\hat{S}, \hat{T}^2)$ as the modular data of a super-MTC.

For a given super-MTC with modular data $(\hat{S}, \hat{T}^2)$, define the following quantities. First,

$$T^{\mathrm{Ind}} \coloneqq \begin{pmatrix} 0 & \hat{T}^2 & 0 \\ \mathbb{1} & 0 & 0 \\ 0 & 0 & (\hat{S}\hat{T}^2)^{-1} \end{pmatrix}. \quad (4)$$

Second, N_c (called the "level candidate"), the smallest positive integer such that every eigenvalue λ_i of T^{Ind} satisfies $\lambda_i^{N_c} = 1$.

Third,

$$\hat{H}(n) \coloneqq \hat{S}^2 \hat{T}^{n^2-n} \hat{S} \hat{T}^{-(\bar{n}-1)} \hat{S} (\hat{T}^2 \hat{S})^{n-1} \quad (5)$$

for each $n \in \mathbb{Z}_{N_c}^{\times}$, the multiplicative group of integers modulo N_c. $\bar{n}$ denotes the modular inverse of n, i.e., an element of $\mathbb{Z}_{N_c}^{\times}$ such that $n\bar{n} \equiv 1 \pmod{N_c}$.

Theorem 1—A super-MTC with modular data $(\hat{S}, \hat{T}^2)$ has $c = 0 \bmod \frac{1}{2}$ if and only if the set of equations

$$\begin{aligned} \hat{T}^N &= \mathbb{1} \\ \hat{S}^2 &= \hat{H}(-1) \\ \hat{H}(n_1)\hat{H}(n_2) &= \hat{H}(n_1 n_2) \\ \hat{S}\,\hat{H}(n) &= \hat{H}(\bar{n})\hat{S} \end{aligned} \quad (6)$$

are satisfied for each $n, n_1, n_2 \in \mathbb{Z}_{N_c}^{\times}$. (The first equation is well-defined since N_c is even).

Theorem 2—A super-MTC with modular data $(\hat{S}, \hat{T}^2)$ such that $c = 0 \bmod \frac{1}{2}$ admits a gapped boundary only if

$$\sum_a \hat{H}(n)_{1a} = +1 \quad (7)$$

for all $n \in \mathbb{Z}_{N_c}^{\times}$.

Here, the index 1 denotes the vacuum component. $\hat{H}(n)$ is always a signed permutation matrix, so $\sum_a \hat{H}(n)_{1a}$ simply picks out the single nonzero entry of the first row of $\hat{H}(n)$.

The two theorems imply that, given $(\hat{S}, \hat{T}^2)$ of a super-MTC, we can test whether they are compatible with a gapped boundary by simply computing N_c and all $\hat{H}(n)$, and seeing whether Eqs. (6) and (7) are satisfied. This process is completely algorithmic.

In the rest of the Letter, we will prove these two theorems using the representation-theoretic structure of the modular data of super-MTCs and their modular extensions, and then discuss examples.

Representation theory and the chiral central charge—The key idea for proving Theorem 1 comes from observing that the chiral central charge can be thought of as an obstruction for the modular data forming a linear representation. First, consider the bosonic case. It is well-known that the modular data (S, T) form a projective representation ρ [called the "modular representation" of the modular group $\mathrm{SL}_2(\mathbb{Z})$]. $\mathrm{SL}_2(\mathbb{Z})$ is generated by

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (8)$$

These generators satisfy the relation (among others)

$$(st)^3 = s^2. \quad (9)$$

The representation ρ can be specified by the representation matrices for the two generators

$$S = \rho(s), \qquad T = \rho(t). \quad (10)$$

Hence, we will often refer to the pair of generating matrices as a representation. The representation (S, T) is only projective in general, i.e., it only satisfies the relation up to a phase, and this depends on the chiral central charge c,

$$(ST)^3 = e^{2\pi i c/8} S^2. \tag{11}$$

The representation ρ generated by (S, T) is linear if and only if $c = 0 \bmod 8$; hence, we can determine whether $c = 0 \bmod 8$ by testing whether (S, T) forms a linear representation. We also note that the representation given by

$$S_{\mathrm{CFT}} = S, \qquad T_{\mathrm{CFT}} = e^{-2\pi i c/24} T \tag{12}$$

is linear, for any c consistent with the value of c mod 8 determined from Eq. (11) [33,34].

In the fermionic case, the modular data $(\hat{S}, \hat{T}^2)$ forms a projective representation $\hat{\rho}$ of Γ_θ, the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by s and t^2 [17]. Γ_θ has no relation like Eq. (9); thus, we cannot use Eq. (11) to determine its chiral central charge.

The modular representation ρ, however, satisfies more relations. The modular representation is in fact a *congruence representation* of $\mathrm{SL}_2(\mathbb{Z})$ [35], which means the kernel of ρ contains $\Gamma(N)$ (for some positive integer N), the principal congruence subgroup of level N of $\mathrm{SL}_2(\mathbb{Z})$, defined as

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}. \tag{13}$$

The smallest N for which $\ker \rho \geq \Gamma(N)$ is satisfied is called the level of ρ, and N is equal to N_{FS}, the order of $T \coloneqq \rho(t)$. [33,35]. Equivalently, congruence representations can be defined as those representations of $\mathrm{SL}_2(\mathbb{Z})$ that can also be thought of as representations of $\mathrm{SL}_2(\mathbb{Z})/\Gamma(N) \simeq \mathrm{SL}_2(\mathbb{Z}_N)$.

The modular Γ_θ-representation $\hat{\rho}$ of a super-MTC is also congruence [36], and hence satisfies a large number of additional relations. Explicitly, these relations can be written in terms of $\hat{H}(n) \coloneqq \hat{\rho}\left(\begin{smallmatrix} n & 0 \\ 0 & \bar{n} \end{smallmatrix}\right)$ [equivalent to Eq. (5)], as Eq. (6), where N replaces N_c [14,37]. The main idea, then, is to use these relations as a test of $c = 0 \bmod \frac{1}{2}$ of a given super-MTC: these relations will be satisfied exactly if $\hat{\rho}$ is linear, but only up to a phase if $\hat{\rho}$ is projective.

Unlike in the bosonic case, however, the level N of $\hat{\rho}$ cannot be fixed from the order of $\hat{T}^2$ alone. Since the relations Eq. (6) depend on N, we need a way to determine the candidate level N_c for which we will test Eq. (6).

Induced representations and the level—Given a super-MTC $\mathcal{B}$ with modular data $(\hat{S}, \hat{T}^2)$, the modular extension $\breve{\mathcal{B}}$ has modular representation that, in an appropriate basis, takes the block-diagonal form [17,36]

$$\rho(s) = \begin{pmatrix} \hat{S} & 0 & 0 & 0 \\ 0 & 0 & 2A & \sqrt{2}X \\ 0 & 2A^T & 0 & 0 \\ 0 & \sqrt{2}X^T & 0 & 0 \end{pmatrix} \oplus B,$$

$$\rho(t) = \begin{pmatrix} 0 & \hat{T} & 0 & 0 \\ \hat{T} & 0 & 0 & 0 \\ 0 & 0 & \hat{T}_v & 0 \\ 0 & 0 & 0 & T_\sigma \end{pmatrix} \oplus \hat{T}_v. \tag{14}$$

The first block, which we denote by ρ_+, itself takes the block form

$$\rho_+(s) = \begin{pmatrix} \hat{S} & 0 & 0 \\ 0 & 0 & C \\ 0 & C^T & 0 \end{pmatrix}, \quad \rho_+(t) = \begin{pmatrix} 0 & \hat{T} & 0 \\ \hat{T} & 0 & 0 \\ 0 & 0 & T_{\text{R-NS}} \end{pmatrix}, \tag{15}$$

where $C = \left(2A\ \sqrt{2}X\right)$ and $T_{\text{R-NS}} = \left(\begin{smallmatrix} \hat{T}_v & 0 \\ 0 & T_\sigma \end{smallmatrix}\right)$ are square matrices with the same dimension as $\hat{S}$ and $\hat{T}$. After restricting to Γ_θ this is further reducible and has $(\hat{S}, \hat{T}^2)$ in the first block.

When $\breve{\mathcal{B}}$ has $c = 0 \bmod 8$, the modular representation ρ is a linear representation of $\mathrm{SL}_2(\mathbb{Z})$, and hence ρ_+ is also linear. Thus, the Γ_θ-representation $\hat{\rho}$ obtained by restriction of ρ_+ is also linear. On the other hand, given a linear Γ_θ representation generated by $(\hat{S}, \hat{T}^2)$, we can form a representation of $\mathrm{SL}_2(\mathbb{Z})$ *induced* from it as [14]

$$\rho^{\mathrm{Ind}}(s) = \begin{pmatrix} \hat{\rho}(s) & 0 & 0 \\ 0 & 0 & \hat{\rho}(s)^2 \\ 0 & \mathbb{1} & 0 \end{pmatrix},$$

$$\rho^{\mathrm{Ind}}(t) = \begin{pmatrix} 0 & \hat{\rho}(t)^2 & 0 \\ \mathbb{1} & 0 & 0 \\ 0 & 0 & (\hat{\rho}(s)\hat{\rho}(t)^2)^{-1} \end{pmatrix}. \tag{16}$$

We now state two lemmas, whose proofs are included as Appendixes A and B, respectively.

Lemma 1—The induced representation ρ^{Ind}, Eq. (16), is isomorphic to ρ_+, Eq. (15).

Corollary 1—Given a $c = 0 \bmod \frac{1}{2}$ super-MTC $\mathcal{B}$ with modular data $(\hat{S}, \hat{T}^2)$, the $c = 0 \bmod 8$ modular extension $\breve{\mathcal{B}}$ has T of order N_c, where N_c is the smallest positive integer such that every eigenvalue λ_i of T^{Ind} satisfies $\lambda_i^{N_c} = 1$.

Proof—As a consequence of Lemma 1, $T^{\mathrm{Ind}} \coloneqq \rho^{\mathrm{Ind}}(t)$ and $T_+ \coloneqq \rho_+(t)$ have the same set of eigenvalues. T of $\breve{\mathcal{B}}$ is equivalent to $T_+^{\mathrm{diag}} \oplus \hat{T}_v$ where T_+^{diag} is the diagonal matrix consisting of the eigenvalues of T_+. $\hat{T}_v$ simply duplicates a part of T_+^{diag} [see Eq. (14)], so the set of distinct eigenvalues of T and T_+ are the same. Thus, N_c is the order of T. ∎

Lemma 2—The level N of the modular representation $\hat{\rho}$ of a super-MTC $\mathcal{B}$ is equal to the level of ρ, the modular representation of the $c=0$ modular extension $\breve{\mathcal{B}}$.

Now we are ready to prove Theorem 1.

Proof of Theorem 1—If $c = 0 \bmod \frac{1}{2}$, there exists a modular extension with $c = 0 \bmod 8$. The corresponding modular representation is linear since $c = 0 \bmod 8$, and has level N_{FS}, which, by Corollary 1, is given by N_c, the order of the eigenvalues of T^{Ind}.

By Lemma 2, the level of $\hat{\rho}$ is equal to the level of ρ, which is N_{FS}. Thus, $\hat{\rho}$ is a Γ_θ-congruence representation of level $N_{\mathrm{FS}} = N_c$, which means it satisfies Eq. (6).

Conversely, suppose $c \neq 0 \bmod \frac{1}{2}$. Then, if (S,T) denotes the modular data of an arbitrary modular extension, $(S, e^{-2\pi i c/24}T)$ generates a linear representation of $\mathrm{SL}_2(\mathbb{Z})$ [see Eq. (12)]. According to the proof of Theorem III.1 of Ref. [14], this restricts to a linear Γ_θ-congruence representation

$$(\hat{S}, e^{-2\pi i c/12}\hat{T}^2). \tag{17}$$

This differs from the representation formed by the modular data $(\hat{S},\hat{T}^2)$ by tensor product with a one-dimensional Γ_θ-representation χ given by

$$\chi(s) = 1, \qquad \chi(t^2) = e^{-2\pi i c/12}. \tag{18}$$

Hence, $(\hat{S},\hat{T}^2)$ is linear if and only if χ is a linear congruence representation of Γ_θ. Linear congruence representations can be classified using the method of Ref. [14], and 1D irreducible representations of the form $(1, e^{-2\pi i c/12})$ exist precisely for $c = 0 \bmod \frac{1}{2}$ [38]. So, for $c \neq 0 \bmod \frac{1}{2}$, $(\hat{S},\hat{T}^2)$ is not a linear congruence representation, and hence does not satisfy the conditions Eq. (6) linearly, for any level.

Thus, $c = 0 \bmod \frac{1}{2}$ if and only if Eq. (6) is satisfied for N. ∎

Galois conjugates and higher central charges—Before moving on to the proof of Theorem 2, we note that an MTC has "Galois conjugates", that is, MTCs with different but related modular data. The higher central charges of a given MTC $\mathcal{C}$ can be computed as the chiral central charge of Galois conjugate MTCs $\mathcal{C}'$ [11]. In particular, when $c = 0 \bmod 8$, a Galois conjugate exists for each $n \in \mathbb{Z}_{N_{FS}}^\times$, and the Galois conjugate S matrix is given by

$$S'(n) = H(n)S, \tag{19}$$

where

$$H(n) := \rho\begin{pmatrix} n & 0 \\ 0 & \bar{n} \end{pmatrix} = S^2 T^{n^2-n} S T^{-(\bar{n}-1)} S (T^2 S)^{n-1}. \tag{20}$$

$H(n)$ is always a signed permutation matrix [33,39,40].

According to Ref. [11] [Eqs. (83) and (84)], when $c = 0 \bmod 8$, the higher central charges ξ_n can be computed as the phase of $S'(\bar{n})_{11}$,

$$\xi_n = \frac{S'(\bar{n})_{11}}{|S'(\bar{n})_{11}|}. \tag{21}$$

We now state the following lemma (proved in Appendix C):

Lemma 3—$\hat{H}(n)$ is a signed permutation matrix, and $\sum_a \hat{H}(n)_{1a} = \sum_a H(n)_{1a}$.

Proof of Theorem 2—First,

$$S'(\bar{n})_{11} = \sum_a H(\bar{n})_{1a} S_{a1} = \sum_a H(\bar{n})_{1a} \frac{d_a}{D}. \tag{22}$$

Note that d_a and D are real numbers. Assuming that the MTC we start out with is unitary, so that $d_a > 0$, we have, from Eq. (21),

$$\xi_n = \sum_a H(\bar{n})_{1a}, \tag{23}$$

whose values are ± 1.

Now, for a super-MTC $\mathcal{B}$ with $c = 0 \bmod \frac{1}{2}$, the ξ_n of the $c = 0 \bmod 8$ modular extension $\breve{\mathcal{B}}$ obey the above. We can restrict the $\mathrm{SL}_2(\mathbb{Z})$ representation ρ formed by the modular data of $\breve{\mathcal{B}}$ to Γ_θ, after which ρ becomes reducible with $\hat{\rho}$ as the first block. $H(\bar{n})$ survives the restriction, and is thus block-diagonalizable, with $\hat{H}(\bar{n})$ as the first block.

By Lemma 3, $\hat{H}(\bar{n})$ is again a signed permutation matrix, and the nonzero entry of the first row of $\hat{H}(\bar{n})$ is equal to the nonzero entry of the first row of $H(\bar{n})$.

Hence, we can compute the higher central charges of the modular extension $(\breve{\mathcal{B}}, 0)$ purely in terms of the modular data of the super-MTC $\mathcal{B}$ as

$$\xi_n = \sum_a \hat{H}_{1a}(\bar{n}). \tag{24}$$

Since admitting a gapped boundary requires $\xi_n = +1$ for all $n \in \mathbb{Z}_{N_{\mathrm{FS}}}^\times$, and $\bar{n}$ is in $\mathbb{Z}_{N_{\mathrm{FS}}}^\times$ if and only if n is, this completes the proof of Theorem 2. ∎

Examples—We apply our test to known examples of super-MTCs. Our test can be implemented easily on *Mathematica*, for example. We only consider intrinsically fermionic topological orders (corresponding to nonsplit super-MTCs), which are not obtained from simply stacking a bosonic topological order with the trivial fermionic topological order. Reference [14] classified super-MTCs up to rank 10. There are many super-MTCs with $c = 0 \bmod \frac{1}{2}$; however, we find that only the following (among unitary super-MTCs) pass the higher central charge test: (1) $PSU(2)_{10}$ and $PSU(2)_{-10}$ (rank 6) and (2) $PSU(2)_6 \boxtimes_f PSU(2)_6$ (rank 8). In particular, the new classes of rank 10 modular data found by Ref. [14], which were constructed using the Drinfeld center of near-group fusion categories in Ref. [41], do not admit gapped boundaries in spite of having $c = 0 \bmod \frac{1}{2}$.

Reference [16] also lists several super-MTCs of rank 12 and 14. Among these, we test those super-MTCs that are nonsplit with $c = 0 \bmod \frac{1}{2}$. For rank 12, one class of examples come from the fermion condensation of $U(1)_8 \boxtimes \mathrm{Ising}^{\nu}$ or similar [42]. These do not have vanishing ξ_n. For rank 12, another class of examples come from fermion condensation of $(B_2)_2 \boxtimes U(1)_4$ or similar. These do not have vanishing ξ_n. For rank 14, we have a class of examples that come from the fermion condensation of $\mathrm{Ising}^{\nu_1} \boxtimes \mathrm{Ising}^{\nu_2} \boxtimes \mathrm{Ising}^{\nu_3}$. These have vanishing ξ_n.

Moreover, $\mathrm{SU}(2)_{4k+2}$ and $\mathrm{SO}(4k+2)_2$ are known to be infinite series of spin-MTCs, so their fermion condensations yield super-MTCs [45]. For $\mathrm{SU}(2)_{4k+2}$, $c \neq 0 \bmod \frac{1}{2}$ (except for $\mathrm{SU}(2)_{10}$, which was considered earlier), so the existence of a gapped boundary is already obstructed by the chiral central charge. For $\mathrm{SO}(4k+2)_2$, we check up to $4k+2 = 68$ (we use Refs. [46,47] to compute the modular data). we find that only the fermion condensation of $\mathrm{SO}(36)_2$ (rank 14) has vanishing higher central charges.

We may ask whether this topological order actually admits a gapped boundary. This super-MTC, contains three bosonic simple objects, denoted by $X_{\mathbf{0}}, X_{2\lambda_1}, X_{\gamma^{12}}$ (we use the notation of Ref. [47]), with quantum dimensions 1,1, and 2, respectively. There are also three simple objects with $\theta_a = -1$, which comes from fusing the above objects with the fermion $X_{2\lambda_{17}}$. We can form the NS-sector Lagrangian algebra object

$$\mathcal{L}_{\mathrm{NS}} = X_{\mathbf{0}} \oplus X_{\lambda_1} \oplus X_{2\lambda_{17}} \oplus X_{2\lambda_{18}} \oplus 2X_{\gamma^6} \oplus 2X_{\gamma^{12}} \qquad (25)$$

(see the Supplemental Material [23] or Ref. [48] for an explanation of fermionic Lagrangian algebras). If we write this as $\mathcal{L}_{\mathrm{NS}} = \bigoplus_{a \in \mathrm{Obj}(\mathcal{B})} Z_a a$, the coefficient vector Z_a is invariant under $\hat{S}$ and $\hat{T}^2$ of $\mathcal{B}$, which provides strong evidence that this forms a gapped boundary.

In addition, for cases where the explicit modular extension data is available (e.g., via Ref. [29]), we verify that higher central charge computed via our method from $(\hat{S}, \hat{T}^2)$ agrees with the higher central charges computed from the $c = 0$ modular extension via the bosonic formula Eq. (2).

Discussion—In this Letter, we have developed a method that tests whether the bulk data of a given fermionic topological order is compatible with a gapped boundary. For a given bulk data, expressed as the modular data $(\hat{S}, \hat{T}^2)$ of a super-MTC, we test (1) whether the chiral central charge $c = 0 \bmod \frac{1}{2}$ and, (2) if so, whether the higher central charges of the modular extension vanish. We have applied our test to known examples of super-MTCs, and found that most of them are ruled out from admitting a gapped boundary.

Expressions similar to the higher central charge [Eq. (2)] have been previously used for super-MTCs to investigate properties of fermionic topological orders, such as the gappability of the boundary in the presence of $U(1)$ symmetry [48] and the detection time-reversal invariance [49]. However, the present Letter addresses the long-standing question regarding the constraints on the chiral central charge and gappability of the boundary for general super-MTCs, representing the first practical step toward a fermionic generalization of the Gauss-Milgram formula [Eq. (1)]. This method has relied crucially on the congruence property of the representations formed by the modular data. A deeper investigation of the congruence representation conditions may lead to an algorithm that determines $c \bmod \frac{1}{2}$ even when it is nonzero.

We note that an anomaly indicator-type formula that computes the chiral central charge mod $\frac{1}{2}$ from the super-MTC data should *in principle* exist: it comes from the $K3$ partition function of the 4D spin-Crane-Yetter topological quantum field theory built from the super-MTC. In practice, however, this partition function is difficult to compute and the formula is unknown [50]. Our test, on the other hand, is highly practical.

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[13] For Abelian MTCs, more general values of n are known to give obstructions to the existence of a gapped boundary, and

in fact taken together they form a sufficient condition for the existence of a gapped boundary [11]. In this Letter, we focus on the general non-Abelian case, and only consider ξ_n for n coprime to N_{FS}. Indeed, the Abelian case will not be of much interest for us, since Abelian fermionic topological orders are never "intrinsically fermionic": they can always be obtained by stacking a bosonic topological order with the trivial fermionic topological order [14,15].
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End Matter

Appendix A: Proof of Lemma 1 [51]—The proof relies on Frobenius reciprocity between restriction and induction of representations. We refer to Ref. [14] for an exposition of restriction and induction of representations, as well as Frobenius reciprocity, in the context of $\mathrm{SL}_2(\mathbb{Z})$ and its index-3 subgroup Γ_θ.

Consider ρ_+ from Eq. (15). ρ_+ satisfies two crucial properties: (1) ρ_+ takes a block form, where it acts on three subspaces V_1, V_2, V_3 by mapping among them in a particular way, and (2) if we restrict this to Γ_θ, the first block becomes decomposable and gives us exactly $\hat{\rho}$.

If ρ_+ is irreducible, then Frobenius reciprocity tells us immediately that it must be the induced representation of $\hat{\rho}$, and hence isomorphic to ρ^{Ind}.

If instead ρ_I is a direct sum $\bigoplus_i (\rho_+)_i$ of irreducible representation $(\rho_+)_i$, each of the summands $(\rho_+)_i$ also has to map among the three subspaces, and hence are of dimension $3k_i$, $k_i \in \mathbb{N}$. If we restrict to Γ_θ, $\rho_+|_{\Gamma_\theta}$ and hence each $(\rho_+)_i|_{\Gamma_\theta}$ leaves the first subspace V_1 invariant. Each of $(\rho_+)_i|_{\Gamma_\theta}$ then contributes a k_i-dimensional Γ_θ-representation $\hat{\rho}_i$ to the Γ_θ-representation $\hat{\rho}$ which acts on V_1.

Each $\hat{\rho}_i$ has to be an irreducible representation because, by a straightforward application of Frobenius reciprocity, the restriction of a G-representation R to H can only contain H irreducible representation of dimension greater than or equal to $\{1/[G:H]\}\dim R$; in our case the index is $[\mathrm{SL}_2(\mathbb{Z}):\Gamma_\theta] = 3$). Hence,

$$\hat{\rho} = \bigoplus_i \hat{\rho}_i. \tag{A1}$$

Now, since each $\hat{\rho}_i$ and $(\rho_+)_i$ are irreducible representations, we get $\mathrm{Ind}\hat{\rho}_i = (\rho_+)_i$ by Frobenius reciprocity. Then,

$$\mathrm{Ind}\hat{\rho} = \bigoplus_i \mathrm{Ind}\hat{\rho}_i = \bigoplus_i (\rho_+)_i = \rho_+, \tag{A2}$$

which shows that ρ_+ is indeed the induced representation of $\hat{\rho}$, and hence isomorphic to ρ^{Ind}. ∎

Appendix B: Proof of Lemma 2—Note that the modular data (S,T) of $(\breve{\mathcal{B}},0)$ form a linear representation ρ of $\mathrm{SL}_2(\mathbb{Z})$ with $\ker\rho \geq \Gamma(N_{\mathrm{FS}})$, where $N_{\mathrm{FS}} \coloneqq \mathrm{ord}T$. By the proofs of Theorem 3.1 of Ref. [36]

$$\ker\hat{\rho} \geq \Gamma(N_{\mathrm{FS}}), \tag{B1}$$

i.e., the level of $\hat{\rho}$ is at most N_{FS}.

On the other hand, by Corollary 1, $\rho^{\mathrm{Ind}}(t)$ has order N_{FS}, and thus ρ^{Ind} has congruence level N_{FS}.

$\hat{\rho}$ is a Γ_θ-congruence representation of level m for some m. We can think of $\hat{\rho}$ as a representation of $\Gamma_\theta/\Gamma(m)$ for some even m, and the induced representation ρ^{Ind} as a representation of $\mathrm{SL}_2(\mathbb{Z})/\Gamma(m)$. Then it is clear that $\ker\rho^{\mathrm{Ind}} \geq \Gamma(m)$. But m cannot be any smaller than N_{FS} (note that both m and N_{FS} are even). Hence, the level of $\hat{\rho}$ is at least N_{FS}.

Together, we see that the level of $\hat{\rho}$ is exactly N_{FS}. ∎

Appendix C: Proof of Lemma 3—Let us write

$$H(n) = \begin{pmatrix} A & B & \cdots \\ C & D & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{C1}$$

where A, B, C, D are each d-dimensional matrices (where d is the dimension of $\hat{S}$ and $\hat{T}^2$), and $B = C^T$, $A^T = A$, $D^T = D$ because $H(n)$ is symmetric. As we can block-diagonalize S and T, we can block-diagonalize $H(n)$ with the same basis change, which we denote U. It takes the form

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} \oplus \ldots, \tag{C2}$$

where $\mathbb{1}$ denotes the d-dimensional identity matrix. Then, we have

$$\begin{aligned} &\begin{pmatrix} \hat{H}(n) & 0 & \cdots \\ 0 & \ddots & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \\ &= UH(n)U^\dagger \\ &= \frac{1}{2}\begin{pmatrix} A+B+C+D & A+C-(B+D) & \cdots \\ A-C+B-D & A-C-(B-D) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \end{aligned} \tag{C3}$$

From this we get $\hat{H}(n) = \frac{1}{2}(A+B+C+D)$ and $0 = A+C-B-D = A-C+B-D$, which in turn leads to

$$B = C \tag{C4}$$

and

$$A = D. \tag{C5}$$

Then,

$$\hat{H}(n) = A + B. \tag{C6}$$

Now, we note that A, B are part of $H(n)$, a signed permutation matrix, on the same block row. If an entry $A_{ij} = \pm 1$, then the every entry of $H(n)$ on the same row should be 0—in particular, $B_{ik} = 0$ for all k. It is also clear that if $A_{ij} = \pm 1$, $A_{ik} = 0$ for all $k \neq j$. Then, $A + B$ has no row with multiple nonzero entries.

Similarly, since $C = B$, we can repeat the same analysis for columns, and show that $A + B$ has no column with multiple nonzero entries. Moreover, since A_{ij} and B_{ij} cannot both be nonzero for any i, j, every entry of $A + B$ is 1, 0, or -1.

We know *a priori* that $\hat{H}(n)$ must be unitary. Hence, $\hat{H}(n) = A + B$ is a unitary matrix whose entries are 1, 0, or -1 and with at most one nonzero entry per each row or column, i.e., it is a signed permutation matrix.

Moreover, it immediately follows that the first row of $A + B$ contains a nonzero entry ± 1, and this value must equal the value of the nonzero entry of the first row of $H(n)$. ∎