Decoding the Drive-Bath Interplay: A Guideline to Enhance Superconductivity

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Driven-dissipative physics lie at the core of quantum optics. However, the full interplay between a driven quantum many-body system and its environment remains relatively unexplored in the solid state realm. In this Letter, we inspect this interplay beyond the commonly employed stroboscopic Hamiltonian picture based on the specific example of a driven superconductor. Using the Shirley-Floquet and Keldysh formalisms as well as a generalization of the notion of superconducting fitness to the driven case, we show how a drive which anticommutes with the superconducting gap operator generically induces an unusual particle-hole structure in the spectral functions from the perspective of the thermal bath. Concomitant with a driving frequency which is near resonant with the intrinsic cutoff frequency of the underlying interaction, this spectral structure can be harnessed to enhance the superconducting transition temperature. Our work paves the way for further studies for driven-dissipative engineering of exotic phases of matter in solid-state systems.

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In the past decade, controllable light-matter coupling and Floquet engineering have emerged as powerful tools to tailor a plethora of phenomena in quantum many-body systems. This has permitted the exploration of novel out-of-equilibrium physics in the realm of quantum simulation as well as solid-state platforms. In the context of superconductivity, these tools have been shown to enhance [1–5] or induce superconductivity [6–9], and even generate exotic orders. These include nontrivial topology [10–13], odd-frequency correlations [14,15], η-pairing [16–19], entropy-cooling mechanism [20], as well as Ampèrean [21] and chiral superconductivity [22]. Most of these Floquet engineering schemes rely on the effective stroboscopic Hamiltonians [23–25], which are renormalized, acquire new terms, or obtain an extra synthetic dimension [26–29].

Meanwhile, dissipative environments are ubiquitous, and help mitigate the problem of heating endemic to most interacting driven systems. Dissipation has also been developed as a resource to engineer correlated steady states especially for quantum computation applications [30]. Combining both drive and thermal dissipation paves the way for the exploration of surprising phenomena associated with the *relative rotation* between the system and the bath, which is intrinsically beyond the scope of effective stroboscopic Hamiltonians. In quantum optical-gaseous systems, dissipation is well captured by the rotating wave approximation and thus the Lindblad formalism, as the typical system energy scales (<GHz) are much smaller than the driving frequency (~THz) [31,32]. This formalism predicts unexpected driven-dissipative effects such as dissipative freezing [33], quantum synchronization [34], modified critical behaviors [35–37] and a stability towards high-energy steady states [38–42].

In contrast, the intrinsic energy scales of solid-state systems (~100 GHz) are comparable to the terahertz driving frequencies in state-of-the-art experiments, making the Lindbladian approach insufficient. A more general description of the complex interplay between the drive and the thermal bath requires a combination of the Floquet [23,24] and Keldysh [31,43–45] formalisms.

In this Letter, we explore how this interplay affects long-range order. Focusing on the example of driven-dissipative superconductors, we address whether periodic driving enhances or reduces superconducting order. For static superconductors, the fitness criterion based on the commutator of the normal state Hamiltonian $\hat{H}_{0,\mathbf{k}}$ and the gap matrix $\hat{\Delta}_{\mathbf{k}}$ quantifies the potential stability of superconducting states [46–48]. It motivates us to propose a similar measure for the driven system:

$$[\hat{H}_{\pm 1,\mathbf{k}}, \hat{\Delta}_{\mathbf{k}}]_{-\zeta} = 0, \tag{1}$$

where $\hat{H}_{\pm 1,\mathbf{k}}$ are the lowest order Fourier components of the drive Hamiltonian and $\zeta = +1$ ($\zeta = -1$) denotes the commutator (anticommutator). Surprisingly, anticommuting drives reorder the particle-hole structure of the Floquet spectral functions, as depicted in Fig. 1(a). When coupled to a thermal bath, this structure suggests a general scheme for the enhancement of the superconducting transition

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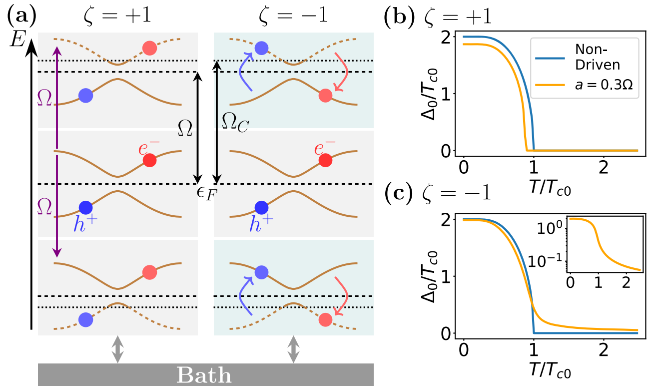

FIG. 1. (a) Schematic representations of the Floquet band structure of the driven superconductor, when the drive (left) commutes and (right) anticommutes with the gap, cf. Eq. (1). In the anticommuting case, the roles of particles and holes are exchanged in every alternating temporal Brillouin zone. This ensures the excitations unfavorable to superconductivity lie outside of the cutoff frequency Ω_C, and, consequently, the system is less susceptible to temperature change of the bath. (b),(c) The time-averaged gap Δ_0 as a function of temperature T for (b) $\zeta = +1$ $[H_1 = (a/2)\tau_0\sigma_y]$ and (c) $\zeta = -1$ $[H_1 = (a/2)\mathrm{sgn}(k_x)\tau_z\sigma_y]$. In panel (c), Δ_0 is also plotted in logarithmic scale in the inset. Here, Eq. (9) is solved for flat-band superconductors Eq. (20).

temperature, see Fig. 1(c). Commuting drives, on the other hand, are generically detrimental to superconductivity. Our proposal is rather general and goes beyond standard mechanisms like dynamic squeezing of phonons [1,2] and coherent destruction of electronic tunneling [3–5]. It opens the door for further studies for driven-dissipative engineering of exotic phases of matter in solid-state systems.

We consider a time-periodic single-particle Hamiltonian coupled to a static bath

$$\mathcal{H} = \sum_{\ell\in\mathbb{Z}} e^{i\ell\Omega t} \sum_{\mathbf{k}} \Psi_{\mathbf{k}}^\dagger H_{\ell,\mathbf{k}} \Psi_{\mathbf{k}} + \sum_{\mathbf{q},r} \xi_{\mathbf{q},r} b_{\mathbf{q},r}^\dagger b_{\mathbf{q},r} + \sum_{\mathbf{k},\mathbf{q},s,r} \left(W_{\mathbf{k},\mathbf{q}} c_{\mathbf{k},s}^\dagger b_{\mathbf{q},r} + \mathrm{H.c.} \right), \qquad (2)$$

where $c_{\mathbf{k},s}$ ($b_{\mathbf{q},r}$) are the fermionic annihilation operators of the system (bath) with momenta $\mathbf{k}$ ($\mathbf{q}$) and internal degrees of freedom s (r), and $\Psi_{\mathbf{k}} = (c_{\mathbf{k},s_1}, c_{\mathbf{k},s_2}, \ldots)^{\mathrm{T}}$; $W_{\mathbf{k},\mathbf{q}}$ describes the weak system-bath coupling, while $\xi_{\mathbf{q},r}$ is the bath dispersion. We use the Floquet-Keldysh formalism in this Letter.

The Shirley-Floquet formalism captures the time periodicity of the Hamiltonian by expressing it in the Floquet representation as [24,25,49] ($\hbar = k_B = 1$)

$$\mathbb{H} = \sum_{\ell\in\mathbb{Z}} H_\ell \mathbb{F}_\ell - \Omega\mathbb{N}, \qquad (3)$$

where $(\mathbb{F}_\ell)_{m,n} = \delta_{m,n-\ell}$ and $\mathbb{N}_{m,n} = m\delta_{m,n}$ are matrices in Floquet space, see Supplemental Material [50]. Since Floquet matrix entries (m,n) sharing the same $\ell = m-n$ are physically equivalent, they can be categorized into the ℓth family of the Wigner representation [49] as

$$H_\ell(\omega) \equiv \mathbb{H}_{m,m+\ell}[\omega - (m + \ell/2)\Omega]. \qquad (4)$$

This structure also applies to other quantities as Green's functions $\mathbb{G}$ and thermal distributions $\boldsymbol{\rho}$.

In the Keldysh formalism, the bath is the source of two effects [31,44,45]. First, the bath's spectral function $\Sigma(\omega) \equiv \pi \sum_{\mathbf{q},r} |W_{\mathbf{k},\mathbf{q}}|^2 \delta(\omega - \xi_{\mathbf{q},r})$ enters the system's retarded or advanced Green's function as $\omega \mapsto \omega_\pm = \omega \pm i\Sigma(\omega)$, widening its poles and rendering the excitation lifetime finite. Our results are qualitatively independent of the specific structure of $\Sigma(\omega)$, and we consider a Markovian bath $\Sigma(\omega) = \Sigma$ for an emphasis on the qualitative features. Second, the bath defines the thermal density distribution $\boldsymbol{\rho}$, which relates the retarded and advanced Green's function $\mathbb{G}^{R/A} = (\omega_\pm \mathbb{F}_0 - \mathbb{H})^{-1}$, to the Keldysh Green's function through the generalized fluctuation-dissipation theorem in Floquet space [31]

$$\mathbb{G}^K = \mathbb{G}^R \boldsymbol{\rho} - \boldsymbol{\rho} \mathbb{G}^A. \qquad (5)$$

Particularly, in the lab frame where the bath is static, $\boldsymbol{\rho}$ has a simple form in the Wigner representation as

$$\rho_\ell = \tanh(\omega/2T)\delta_{\ell,0}. \qquad (6)$$

The drive induces an intrinsic relative rotation between the system and the bath, which cannot be eliminated in any rotating frame. A usual strategy to solve the driven system is to diagonalize the system Hamiltonian in Floquet space, $\mathbb{H} \mapsto \mathbb{H}' = \mathbb{P}^\dagger \mathbb{H} \mathbb{P} = H_0' \mathbb{F}_0 - \Omega\mathbb{N}$, where H_0' is the effective stroboscopic Hamiltonian, and the transformation operator $\mathbb{P}$ is a function of $\mathbb{F}_\ell$ [25]. However, the frequency dependence in the Floquet structure of $\boldsymbol{\rho}(\omega)$ makes it noncommuting with both $\mathbb{G}^R$ and $\mathbb{P}$. As a consequence, the bath distribution becomes generally nonthermal in the rotating frame, see Supplemental Material [50],

$$\boldsymbol{\rho} \mapsto \boldsymbol{\rho}' = \mathbb{P}^\dagger \boldsymbol{\rho} \mathbb{P} \neq \boldsymbol{\rho}. \qquad (7)$$

For example, for systems effectively described by a Lindbladian, we obtain a frequency-detuned thermal distribution $\boldsymbol{\rho}'(\omega) = \boldsymbol{\rho}(\omega + \Omega)$, which suggests the inadequacy of the effective stroboscopic Hamiltonian to comprehensively capture the full dissipative physics. More generally, a Floquet Fermi liquid is obtained [51], leading to more exotic effects.

From now on, we focus on the specific example of superconductors in the Nambu spinor basis $\Psi_{\mathbf{k}} = (c_{\mathbf{k}\uparrow}, c_{\mathbf{k}\downarrow}, c_{-\mathbf{k}\uparrow}^\dagger, c_{-\mathbf{k}\downarrow}^\dagger)^{\mathrm{T}}$. The single-particle Hamiltonian of the superconductor $\hat{H}_{\ell,\mathbf{k}}$ entering Eq. (2) is now explicitly written in Nambu (with Pauli matrices τ_i) and spin (σ_i) spaces. A time-independent attractive interaction ($g > 0$) described by

$$\mathcal{H}_{\text{int}} = -\sum_{\mathbf{k}_1,\mathbf{k}_2} g_{\mathbf{k}_1,\mathbf{k}_2} c^{\dagger}_{\mathbf{k}_1\uparrow} c^{\dagger}_{-\mathbf{k}_1\downarrow} c_{\mathbf{k}_2\uparrow} c_{-\mathbf{k}_2\downarrow} \tag{8}$$

induces superconductivity in a specific symmetry channel characterized in the mean-field treatment by the spin-singlet gap matrix $\hat{\Delta}_{\mathbf{k}} = d_{\mathbf{k}}\tau_y\sigma_y$.

To evaluate the impact of drive and dissipation on the superconducting order, we treat the interaction in the mean-field limit using a Hubbard-Stratonovich transformation generalized to the Floquet-Keldysh space [52]. In contrast to Ref. [52], we explicitly consider the effects stemming from Eqs. (5) and (7). Representing the gap in terms of its Fourier components $\Delta(t) = \sum_{\ell\in\mathbb{Z}} \Delta_\ell e^{i\ell\Omega t}$, we obtain a self-consistent gap equation

$$\sum_{\ell\in\mathbb{Z}} \Delta_\ell \mathbb{F}_\ell = ig\int d\omega \sum_{\mathbf{k}} \mathrm{tr}_{\tau\sigma}\left(\hat{\Delta}_{\mathbf{k}} \hat{\mathbb{G}}^{K}_{\mathbf{k}}\right), \tag{9}$$

where the trace is normalized $\mathrm{tr}\mathbf{1} = 1$, and the Keldysh Green's function is defined by Eq. (5), with

$$\left(\hat{\mathbb{G}}^{R/A}_{\mathbf{k}}\right)^{-1} = \omega_\pm \tau_0\sigma_0\mathbb{F}_0 - \hat{\mathbb{H}}_{\mathbf{k}} - \hat{\Delta}_{\mathbf{k}} \sum_{\ell\in\mathbb{Z}} \Delta_\ell \mathbb{F}_\ell. \tag{10}$$

Notably, using $[\mathbb{P}, \mathbb{F}_\ell]_- = 0$ and the invariance of trace under similarity transformations, the gap equation can be shown to be invariant under a general change of reference frame, see Supplemental Material [50]. This indicates that any gap oscillation in a driven-dissipative superconductor is intrinsic. It is a manifestation of Eq. (7) and allows us to work in the lab frame where calculations are significantly simplified.

To investigate the effects of the intrinsic system-bath rotation, we consider a weak, sinusoidal drive with momentum-dependent amplitude $a_{\mathbf{k}}$ implemented upon a superconductor with electronic dispersion $\epsilon_{\mathbf{k}}$,

$$\hat{H}_{0,\mathbf{k}} = \epsilon_{\mathbf{k}}\tau_z\sigma_0, \quad \hat{H}_{\pm1,\mathbf{k}} = \frac{a_{\mathbf{k}}}{2}\tau_\mu\sigma_\nu, \quad \hat{H}_{|\ell|\geq 2} = 0, \tag{11}$$

with $\mu \in \{0, z\}$ and $|a_{\mathbf{k}}| \ll \Omega$. The driving frequency is at least comparable to the electronic energy scale $\Omega \gtrsim \epsilon_{\mathbf{k}}$. The weak drive motivates us to assume *a priori* that the gap is dominated by its time-averaged value $\Delta_0 \gg |\Delta_{|\ell|\geq 1}|$,

$$\hat{\Delta}_{\mathbf{k}} \sum_{\ell\in\mathbb{Z}} \Delta_\ell \mathbb{F}_\ell \approx \Delta_0 d_{\mathbf{k}}\tau_y\sigma_y\mathbb{F}_0, \tag{12}$$

where we have chosen the gauge $\Delta_0 \geq 0$. This approximation captures the dominant features of the driven-dissipative superconductor by allowing us to write a Dyson equation for the Green's function as

$$\hat{\mathbb{G}}^{R}_{\mathbf{k}} = \hat{\mathring{\mathbb{G}}}^{R}_{\mathbf{k}} \sum_{n=0}^{\infty} \left[\frac{a_{\mathbf{k}}}{2}\tau_\mu\sigma_\nu(\mathbb{F}_1 + \mathbb{F}_{-1})\hat{\mathring{\mathbb{G}}}^{R}_{\mathbf{k}}\right]^n. \tag{13}$$

Here, $\hat{\mathring{\mathbb{G}}}^{R}_{\mathbf{k}} \equiv \hat{\mathbb{G}}^{R}_{\mathbf{k}}(a_{\mathbf{k}} = 0)$ describes the nondriven system and can be solved as $(\hat{\mathring{G}}^{R}_{\mathbf{k}})_\ell(\omega) = \delta_{\ell,0}\hat{\mathring{G}}^{R}_{\mathbf{k}}(\omega)$, with

$$\hat{\mathring{G}}^{R}_{\mathbf{k}}(\omega) = \frac{\omega_+\tau_0\sigma_0 + \epsilon_{\mathbf{k}}\tau_z\sigma_0 + \Delta_0 d_{\mathbf{k}}\tau_y\sigma_y}{\omega_+^2 - E_{\mathbf{k}}^2}, \tag{14}$$

where $E_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}^2 + \Delta_0^2 d_{\mathbf{k}}^2}$. By substituting the ansatz Eq. (13) into the gap equation Eq. (9), the Fourier components of the gap Δ_ℓ are determined by the Wigner Green's functions $(\hat{G}^{R}_{\mathbf{k}})_\ell(\omega) \sim (a_{\mathbf{k}}/2)^{|\ell|}\hat{\mathring{G}}^{R}_{\mathbf{k}}(\omega) \times \prod_{n=1}^{\ell} [\tau_\mu\sigma_\nu \hat{\mathring{G}}^{R}_{\mathbf{k}}(\omega \pm n\Omega)]$. By noticing that $a_{\mathbf{k}}\hat{\mathring{G}}^{R}_{\mathbf{k}}(E_{\mathbf{k}} - n\Omega) \sim a_{\mathbf{k}}/\Omega$ for $n \neq 0$, we find $\Delta_\ell \sim (a_{\mathbf{k}}/\Omega)^{|\ell|}\Delta_0$, which validates Eq. (12).

The dominance of the static component of the order parameter indicates that the superconducting excitations can be characterized by the electron-hole spectral function $\mathcal{A}^{\text{eh}}$, the anomalous spectral function $\mathcal{A}^{\text{an}}$, and the anomalous response function $\mathcal{R}^{\text{an}}$ of the $\ell = 0$ Wigner Green's functions,

$$\begin{aligned}
\mathcal{A}^{\text{eh}}_{\mathbf{k}}(\omega) &= -\mathrm{Im}\,\mathrm{tr}_{\tau\sigma}\left[(\hat{G}^{R}_{\mathbf{k}})_{\ell=0}\right]/\pi, \\
\mathcal{A}^{\text{an}}_{\mathbf{k}}(\omega) &= -\mathrm{Im}\,\mathrm{tr}_{\tau\sigma}\left[d_{\mathbf{k}}\tau_y\sigma_y(\hat{G}^{R}_{\mathbf{k}})_{\ell=0}\right]/\pi, \\
\mathcal{R}^{\text{an}}_{\mathbf{k}}(\omega) &= -\mathrm{Im}\,\mathrm{tr}_{\tau\sigma}\left[d_{\mathbf{k}}\tau_y\sigma_y(\hat{G}^{K}_{\mathbf{k}})_{\ell=0}\right]/\pi.
\end{aligned} \tag{15}$$

Notably, the spectral functions capture the driving effects on the superconductor, while the response function captures the thermal effects induced by the bath. The gap equation Eq. (9) is now reduced to

$$\Delta_0 = ig\sum_{\mathbf{k}} \int d\omega \mathcal{R}^{\text{an}}_{\mathbf{k}}(\omega), \tag{16}$$

where Eq. (5) manifests in the lab frame as

$$\mathcal{R}^{\text{an}}_{\mathbf{k}}(\omega) = \tanh\left(\frac{\omega}{2T}\right)\mathcal{A}^{\text{an}}_{\mathbf{k}}(\omega). \tag{17}$$

The driving induces a specific Floquet structure in the spectral functions. For the nondriven system, $(\hat{\mathring{G}}^{R}_{\mathbf{k}})_{\ell=0}$ has two equivalent poles at $\omega_+ = \pm E_{\mathbf{k}}$ reflecting the particle-hole symmetry imposed by the Nambu basis. The corresponding spectral functions are [Figs. 2(a) and 2(d)]

$$\begin{aligned}
\tilde{\mathcal{A}}^{\text{eh}}_{\mathbf{k}}(\omega) &= \frac{1}{2}[L_\Sigma(\omega - E_{\mathbf{k}}) + L_\Sigma(\omega + E_{\mathbf{k}})], \\
\tilde{\mathcal{A}}^{\text{an}}_{\mathbf{k}}(\omega) &= \frac{\Delta_0 d_{\mathbf{k}}}{2E_{\mathbf{k}}}[L_\Sigma(\omega - E_{\mathbf{k}}) - L_\Sigma(\omega + E_{\mathbf{k}})],
\end{aligned} \tag{18}$$

where L_Σ denotes a Lorentzian distribution $L_\Sigma(\omega) = \Sigma/[\pi(\omega^2 + \Sigma^2)]$, which recovers a Dirac delta distribution $L_{\Sigma\to 0^+}(\omega) = \delta(\omega)$ in the weakly dissipative limit.

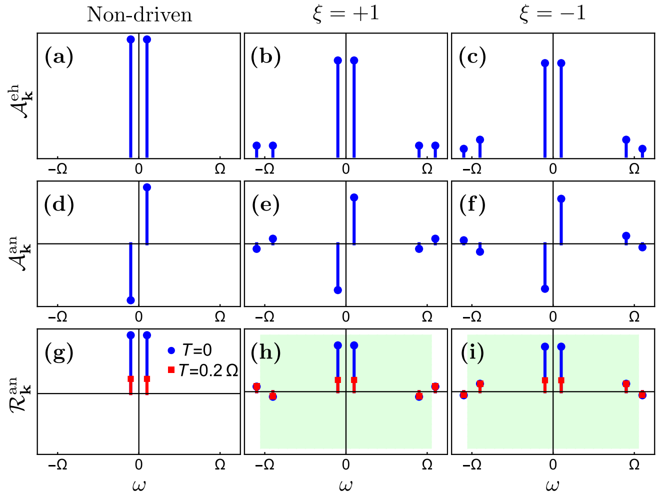

FIG. 2. (a)–(c) The electron-hole spectral functions, (d)–(f) the anomalous spectral functions, and (g)–(i) the anomalous response functions [Eq. (15)] of the weakly dissipative $\Sigma \to 0^+$ superconductors in the first three temporal Brillouin zones $\omega \in (-3\Omega/2, 3\Omega/2]$, for the (a),(d),(g) undriven, (b),(e),(h) commuting ($\zeta = +1$), and (c),(f),(i) anticommuting ($\zeta = -1$) cases. Parameters are taken as $\epsilon_{\mathbf{k}} = 0$, $\Delta_0 d_{\mathbf{k}} = 0.1\Omega$, $a_{\mathbf{k}} = 0.6\Omega$. (g)–(i) For the response functions, results for zero temperature (blue dots) and intermediate temperature (red squares) are shown. The green background indicates the region below the cutoff frequency Ω_C.

In contrast, the Green's function for the driven system $(\hat{G}^R_{\mathbf{k}})_{\ell=0}$ acquires infinitely many shifted poles at $\omega_+ = \pm E_{\mathbf{k}} + n\Omega$, $n \in \mathbb{Z}$. In the high-frequency limit $\Omega \gg \epsilon_{\mathbf{k}}$ for clarity, we find the lowest-order contributions to the spectral functions in the $(n+1)$th temporal Brillouin zone $\omega/\Omega \in (n-1/2, n+1/2]$ behave as [Figs. 2(b), 2(c), 2(e), and 2(f)]

$$\mathcal{A}^{\rm eh}_{\mathbf{k}}(\omega) \approx \sum_{n=-\infty}^{\infty} \frac{1}{(n!)^2}\left(\frac{a_{\mathbf{k}}}{2\Omega}\right)^{2|n|} \tilde{\mathcal{A}}^{\rm eh}_{\mathbf{k}}(\omega - n\Omega),$$

$$\mathcal{A}^{\rm an}_{\mathbf{k}}(\omega) \approx \sum_{n=-\infty}^{\infty} \frac{\zeta^n}{(n!)^2}\left(\frac{a_{\mathbf{k}}}{2\Omega}\right)^{2|n|} \tilde{\mathcal{A}}^{\rm an}_{\mathbf{k}}(\omega - n\Omega). \tag{19}$$

The following discussion is valid for large Ω up to $\Omega \approx 2E_{\mathbf{k}}$, in which case the drive induces a parametric resonance between the particle and hole excitations [50].

We now elucidate the role of the commutator, Eq. (1). In the commuting case [cf. Eq. (1)], $\zeta = +1$, e.g., when the drive is realized by a magnetic field along the y direction $\mu = 0$, $\nu = y$, the Wigner Green's functions admit analytical solutions as $(\hat{G}^R_{\mathbf{k}})_\ell(\omega) = \sum_{\tilde{\ell} \in \mathbb{Z}+\ell/2} \hat{\tilde{G}}^R_{\mathbf{k}}(\omega + \tilde{\ell}\Omega) \times J_{\tilde{\ell}+\ell/2}(a_{\mathbf{k}}/\Omega) J_{\tilde{\ell}-\ell/2}(a_{\mathbf{k}}/\Omega)$, with J_ℓ the Bessel functions of the first kind, see Supplemental Material [50]. The physical consequences of the driving are not reflected in the spectral functions $\int_{-\infty}^{\infty} d\omega \mathcal{A}^{\rm eh/an}_{\mathbf{k}}(\omega) = \int_{-\infty}^{\infty} d\omega \tilde{\mathcal{A}}^{\rm eh/an}_{\mathbf{k}}(\omega)$ [53], but only in the response function $\int_{-\infty}^{\infty} d\omega \mathcal{R}^{\rm an}_{\mathbf{k}}(\omega) < \int_{-\infty}^{\infty} d\omega \tilde{\mathcal{R}}^{\rm an}_{\mathbf{k}}(\omega)$. In this case, the intrinsic system-bath rotation is detrimental to superconductivity, cf. Eq. (16), as confirmed in Fig. 1(b).

In contrast, the anticommuting case, $\zeta = -1$, has more intriguing features. This can be realized by $\mu = z$, $\nu = y$, associated to a Rashba-like spin-orbit coupling. In the anomalous spectral function $\mathcal{A}^{\rm an}$ [Fig. 2(f)], an extra phase shift of π is introduced in the gap $\Delta_0 \to -\Delta_0$ as one moves from one temporal Brillouin zone to the next, which results in an alternating sign in $\mathcal{A}^{\rm an}$ across temporal Brillouin zones. Effectively, the roles of particle and hole are successively exchanged, as depicted in Fig. 1(a). The physical consequences of this phase shifting is evident in the response function $\mathcal{R}_{\rm an}$ [Fig. 2(i)]. Its four poles closest to the Fermi surface at $\omega_+ = \pm E_{\mathbf{k}}$ and $\omega_+ = \pm(E_{\mathbf{k}} - \Omega)$ carry positive weights.

We now discuss the effects of an anticommuting drive on the gap equation, Eq. (16). To the lowest order, the response functions at $\omega_+ = \pm E_{\mathbf{k}}$ contribute to the gap with $(\Delta_0/E_{\rm k})\tanh(E_{\rm k}/2T)$, cf. Eq. (13). This contribution is significant at zero temperature and very sensitive to temperature because of its proximity to the Fermi surface. In contrast, the response functions at $\omega_+ = \pm(E_{\mathbf{k}} - \Omega)$ contribute with $(a^2/4\Omega^2)[\Delta_0(\Omega^2 - 4\epsilon^2_{\mathbf{k}})/E_{\rm k}(\Omega - 2E_{\rm k})^2] \times \tanh[(\Omega - E_{\rm k})/2T]$. Particularly, at large driving frequency $\Omega \gg E_{\mathbf{k}}$, these excitations are distant from the Fermi surface, and their contribution is much less sensitive to temperature. Consequently, the combined contribution from all four excitations is expected to be greater than the non-driven counterpart [Fig. 2(g)] at high temperatures $T > E_{\mathbf{k}}$. To enhance superconductivity, the analysis of the response function motivates us to suppress all other excitations, particularly the ones detrimental to the gap at $\omega_+ = \pm(E_{\mathbf{k}} + \Omega)$. This can be achieved by choosing a drive in close resonance to the intrinsic interaction's cutoff frequency Ω_C, such that $\delta\Omega \equiv \Omega - \Omega_C \ll \Omega$. We assume that the cutoff, as given, for example, by the phonon Debye frequency, remains unaffected by the drive.

We show how our scheme works for the simple scenario of a flat-band system with s-wave superconductivity and a drive associated with a Rashba-like spin-orbit coupling, i.e.,

$$\epsilon_{\mathbf{k}} = 0, \qquad d_{\mathbf{k}} = 1, \qquad a_{\mathbf{k}} = a\,\mathrm{sgn}(k_x). \tag{20}$$

To the order of $O(a^2/\Omega^2)$, the anomalous spectral function can be approximated by

$$\mathcal{A}^{\rm an}(\omega) \approx \left(\frac{1}{2} - \frac{a^2}{4\Omega^2}\right)[L_\Sigma(\omega - \Delta_0) - L_\Sigma(\omega + \Delta_0)] + \frac{a^2}{8\Omega^2}[L_\Sigma(\omega - \Omega + \Delta_0) - L_\Sigma(\omega + \Omega - \Delta_0)], \tag{21}$$

where excitations lying beyond the cutoff frequency $|\omega| > \Omega_C$ are omitted. The corresponding gap equation for Δ_0 in the weakly dissipative limit $\Sigma \to 0^+$ now reads

$$\frac{\Delta_0}{gN} \approx \left(1 - \frac{a^2}{2\Omega^2}\right) \tanh\left(\frac{\Delta_0}{2T}\right) + \frac{a^2}{4\Omega^2} \tanh\left(\frac{\Omega}{2T}\right), \quad \forall\, \Delta_0 > \delta\Omega, \tag{22}$$

where N is the total density of states. Dissipation with characteristic energies comparable to the gap Δ_0 will suppress superconductivity [52].

The enhancement of superconductivity can be confirmed by a finite order parameter beyond the critical temperature of the nondriven superconductor $T_{c0} = gN/2$. At high temperatures $T \gg \Omega$, the gap of the driven superconductor exhibits an asymptotic behavior of $\Delta_0 \sim 1/T$, with a first-order jump from finite to vanishing values at the temperature

$$\frac{T_c}{T_{c0}} \approx \frac{a^2}{8\Omega^2} \frac{\Omega}{\delta\Omega}. \tag{23}$$

This result implies the existence of superconductivity at arbitrarily high temperatures for a resonantly tuned driving frequency $\delta\Omega \to 0$. We confirm our analysis by numerically solving Eq. (9) for flat-band superconductors Eq. (20) at different temperatures; see Fig. 1(c). The numerical solutions confirm our assumption that the oscillating components are negligible compared to the constant component, see Supplemental Material [50]. In the weak coupling limit, the robustness of our scheme manifests itself in its qualitative validity (i) when the driving frequency is set to be resonant with any odd submultiple of the cutoff frequency, and (ii) for dispersive superconductors (see Supplemental Material [50]) and superconducting gap of symmetries other than s wave.

In summary, the Floquet-Keldysh formalism, in conjunction with a generalization of the fitness criterion to driven superconductors, provides a framework to engineer specific drives to enhance the superconducting transition temperature [48]. In the static limit, a normal state fully anticommuting with the gap matrix $[\hat{H}_{\ell=0,\mathbf{k}}, \hat{\Delta}_{\mathbf{k}}]_+ = 0$ induces a maximal gap, which can be reduced by any commutativity. Based on a given static system, a drive in the normal state anticommuting with the gap matrix $[\hat{H}_{\ell=\pm1,\mathbf{k}}, \hat{\Delta}_{\mathbf{k}}]_+ = 0$ further potentially enhances the gap in the vicinity of the transition temperature of the static system. Our mean-field treatment of the interaction mediator intrinsically neglects its fluctuations. These fluctuations potentially stabilize the superconductivity in equilibrium [54–59], but can also induce thermalization and quasiparticle scattering in driven systems. These effects should be further investigated in the future.

We clarify the distinctions between our scheme with others in literature. Schemes for the enhancement of superconducting order based on phonon driving and squeezing [1,2] or coherent suppression of electron tunneling [3–5] rely on the description of stroboscopic Hamiltonians. Moreover, Ref. [60] studied a Rabi drive using the formalism presented here. Unlike the sinusoidal drive discussed here, the Rabi drive inherently induces solely corotating dynamics, validating a simplified treatment using Lindbladians, cf. Eq. (7). Furthermore, the interaction mediator also acts as a thermal bath, and it is usually coupled to the electronic system in a fundamentally different form than Eq. (2). Its effects have been investigated for cavity-mediated [58] and driven phonon-mediated [61,62] superconductors. Particularly in the latter case, features similar to our systems are observed, like first-order transition and the associated hysteresis of the superconducting gap.

Our results indicate that the enhancement of the superconducting transition temperature is most pronounced for flat-band systems [50]. Material platforms with flat electronic bands have been extensively discussed in the context of twisted two-dimensional Van der Waals materials [63,64] and heavy fermion systems [65]. Flat bands have also been theoretically identified near the Fermi level in three-dimensional materials, including Weyl-Kondo semimetals [66] and materials with kagome, pyrochlore, or Lieb sublattice structures [67]. In addition, recent experimental developments have shown that it is possible to generate Floquet bands in Van der Waals materials [68] and graphene [69] with light in the THz regime. These results suggest that the necessary experimental ingredients for realizing our proposal are readily available. Moreover, materials with other unconventional band structures like Van Hove singularities, which has high density of states close to the Fermi surface, can also potentially be candidates of our proposal and manifest more intriguing features [51].

To conclude, inspired by the interplay between drive and dissipation in quantum optical systems, we have illustrated the capability of driven-dissipative engineering, specifically in tailoring superconducting order. This stimulates exploration and generalization of the technique to other ordered states of matter in solid state platforms.

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