Black Hole Formation—Null Geodesic Correspondence

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We provide evidence for a correspondence between the formation of black holes and the stability of circular null geodesics around the collapsing perturbation. We first show that the critical threshold of the compaction function to form a black hole in radiation is well approximated by the critical threshold for the appearance of the first unstable circular orbit in a spherically symmetric background. We also show that the critical exponent in the scaling law of the primordial black hole mass close to the threshold is set by the inverse of the Lyapunov coefficient of the unstable orbits when a self-similar stage is developed close to criticality.

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Introduction—Geodesic motions are crucial in determining the fundamental features of spacetime. Circular geodesics are particularly interesting in this regard. For instance, the binding energy of the last stable circular timelike geodesic in the Kerr geometry may be used to give an estimate of the spin of astrophysical black holes [1–3]. Null unstable geodesics are also intimately linked to the appearance of black holes to external observers and have been associated with the characteristic quasinormal modes of black holes [4–6] which can be thought of as null particles trapped at the unstable circular orbit and slowly leaking out [7–10]. The real part of the quasinormal frequency is set by the angular velocity at the unstable null geodesic, while the imaginary part has been shown to be related to the instability timescale of the orbit [11,12]. Such a timescale is set by the Lyapunov exponent characterizing the rate of separation of infinitesimally close trajectories.

Unstable circular orbits might also help to describe phenomena occurring at the threshold of black hole formation in the high-energy scattering of black holes [13]. Finally, there seems to be a correspondence between the scaling exponent setting the number of orbits of two Schwarzschild black holes before merging into a Kerr black hole and the Lyapunov coefficient of the circular orbit geodesics of the final state Kerr black hole [13], as if the properties of the null geodesics of the final state are connected to the dynamics leading to it.

In this Letter we would like to build upon these results and propose some evidence of a correspondence between the formation of black holes (BHs), specifically in the radiation phase of the early Universe and the properties of the null geodesics around the perturbation which eventually collapse into the BH final state.

We will focus in the radiation phase as we will think of BHs formed in the early Universe, the so-called primordial black holes (PBHs). Indeed, they have become a focal point of interest in cosmology in recent years. In the standard scenario PBHs are formed by the gravitational collapse of sizable perturbations generated during inflation (see Ref. [14] for a recent review). However, our logical path following the physics of null geodesics can be applied to BHs formed in different environments and/or from different fields.

By characterizing the initial perturbation with the corresponding compaction function, we will show that—varying its amplitude—the critical value for which the first circular orbit appears with vanishing Lyapunov coefficient well reproduces the critical value for which a BH is formed. Furthermore, the formation of BHs at criticality is subsequent to a self-similar evolution which results in a final mass following a scaling law with a universal critical exponent [15,16]. We will be able to identify such critical exponent with the inverse of the Lyapunov coefficient of the unstable circular orbits during the self-similar stage of the collapse. Before launching ourselves into the technical aspects, let us set the stage in the next section.

Geodesics stability and Lyapunov exponent—In order to investigate the physics of null geodesics and its relation to the formation threshold of BHs, we find it convenient to work with the metric in the radial gauge and polar slicing (which we will call from now on radial polar gauge).

These coordinates are the generalization of the Schwarzschild coordinates to the nonstatic and nonvacuum spacetime and have been routinely used in the numerical studies of the gravitational collapse resulting in the formation of BHs [15,16]. The metric reads

$$\mathrm{d}s^2 = -\alpha^2(r,t)\mathrm{d}t^2 + a^2(r,t)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{1}$$

Let us consider a physics situation in which the time dependence may be neglected and stationarity can be assumed. Null geodesics are determined by the trajectories which move along the equatorial plane such that

$$-\alpha^2\dot{t}^2 + a^2\dot{r}^2 + r^2\dot{\phi}^2 = 0, \tag{2}$$

where the dots indicate differentiation with respect to the affine parameter and ϕ is the azimuthal angle. Because of the spherical symmetry, one has $\dot{\phi}^2 = L^2/r^4$, where L is the angular momentum. Similarly, stationarity gives $\dot{t}^2 = E^2/\alpha^4$, where E is the conserved energy. The equation of motion can be written as

$$\dot{r}^2 = -V(r) = -\frac{1}{a^2}\left(-\frac{E^2}{\alpha^2} + \frac{L^2}{r^2}\right). \tag{3}$$

A circular orbit at a given radius r_c exists if

$$V(r_c) = V'(r_c) = 0, \tag{4}$$

where the prime indicates differentiation with respect to radial coordinate. These conditions impose, respectively

$$\frac{E^2}{L^2} = \frac{\alpha_c^2}{r_c^2}, \tag{5}$$

$$1 = r_c\frac{\alpha_c'}{\alpha_c}, \tag{6}$$

where the subscript $_c$ means that the quantity in question is evaluated at the radius r_c of the circular null geodesic.

If we slightly perturb the orbit taking $r = r_c + \delta r$ and Taylor expand the potential, we get

$$\delta\dot{r}^2 \simeq -\frac{1}{2}V''(r_c)(\delta r)^2. \tag{7}$$

Writing $\delta\dot{r} = (\partial\delta r/\partial t)\dot{t}$, we obtain

$$\delta r(t) = \delta r(0)e^{\lambda t}, \tag{8}$$

where

$$\lambda = \sqrt{\frac{-V''(r_c)}{2\dot{t}^2(r_c)}} = \frac{1}{a_c}\sqrt{-\alpha_c\alpha_c''} \tag{9}$$

is the Lyapunov coefficient which determines the timescale of the instability of the circular orbits against small perturbations.

One fundamental point to notice is the following. Let us write the condition (6) as $g(r_c) = 0$, where

$$g(r) \equiv 1 - r\alpha'(r)/\alpha(r). \tag{10}$$

If we take the energy associated with the potential for generic timelike orbits we notice that

$$E^2 = \frac{\alpha}{g(r)} = \frac{\alpha^2}{1 - r\alpha'/\alpha} \tag{11}$$

which implies $g(r) > 0$ from the condition that this conserved quantity is indeed the energy and it is real. Furthermore, the condition of lightlike orbits corresponds to the innermost timelike orbit at radius $r = r_c$, implying $g(r_c) = 0$. Since $g(r)$ is positive for timelike orbits, by changing a parameter [which will be identified in Eq. (17) as the amplitude of the compaction function A], one meets the critical radius at which the orbit becomes lightlike. The first time this happens is when the condition $g(r_c) = 0$ is reached at the minimum of $g(r)$, i.e.

$$g_c = g_c' = -r_c\frac{\alpha_c''}{\alpha_c} = 0 \to \lambda = 0. \tag{12}$$

Let us imagine to change the parameter A. Initially no circular orbits are found (red lines of Fig. 2). The first critical value r_c is obtained when the minimum of the function $g(r)$ vanishes, which signals the point where the Lyapunov coefficient vanishes. Further increasing the parameter A, the curve $g(r)$ vanishes for two critical radii (blue lines) for which unstable orbits exist (on the right of the minimum). The position of the stable and unstable orbits can be understood as follows. The potential $V(r)$ (3) in this case goes to $+\infty$ for $r \to 0$, having a minimum closer to $r \to 0$ and a maximum further away as can be seen in Fig. 1. The depth of the minimum is related to the parameter A and it is coincident with the maximum at threshold. As the BH forms the potential will change shape, developing the usual divergence to $-\infty$ for small radii

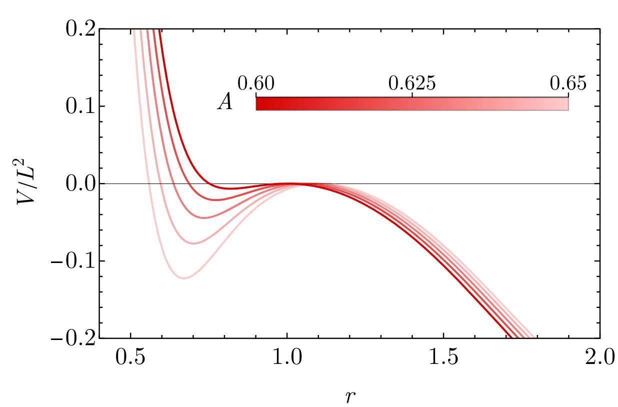

FIG. 1. Plot of the potential obtained from Eq. (3) for multiple values of the parameter A. At threshold we have no maximum or minimum, but an inflection point, while further increasing the values of A give rise to a stable and an unstable circular orbit.

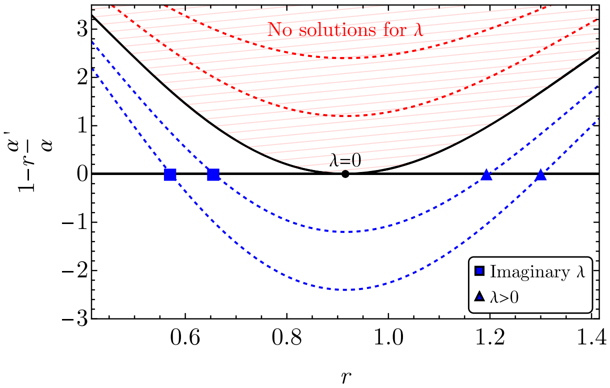

FIG. 2. A schematic representation of the appearance of circular lightlike orbits by increasing amplitude of the compaction function from the red to the blue region.

instead. This change of asymptotic behavior will get rid of the closer minimum, so we can think of the depth of the minimum as the one related to the mass of the future BH, which at threshold gives exactly a zero mass BH, while the outer maximum could be thought of as the one related to the innermost stable circular orbit of the forming BH.

First evidence: The BH threshold from null geodesics—Consider now a perturbation in the radiation energy density which reenters the horizon after having been generated in a previous inflationary stage. Our fundamental assumption is that in the first stage of the dynamics we may consider the fluid to be instantaneously at rest and we can neglect pressure gradients. This assumption is supported by numerical simulations which show that maximum infall radial velocity remains rather small for a time considerably after horizon crossing, till the perturbation has become highly nonlinear [17].

The compaction function in the radial polar gauge is

$$C_{\rm rp}(r) = 1 - \frac{1}{a^2(r)} = \frac{8\pi G_N}{r}\int_0^r {\rm d}x\, x^2\rho(x), \qquad (13)$$

where $\rho(r)$ is the energy density perturbation. (We use the subscript "rp" for the radial polar gauge and "com" for the comoving gauge.) Combining the (rr)- and (tt)-Einstein equations, the lapse function satisfies the following equation during the radiation phase [16]

$$r\frac{\alpha'}{\alpha} = \frac{1}{6}\left(\frac{4C_{\rm rp} + rC'_{\rm rp}}{1 - C_{\rm rp}}\right). \qquad (14)$$

The condition for having a circular orbit therefore becomes

$$10C_{\rm rp}(r_c) + r_c C'_{\rm rp}(r_c) = 6. \qquad (15)$$

This result is already encouraging as it provides values $\mathcal{O}(0.5)$ of the compaction function for which a circular orbit exists, that is in the ballpark of the critical values for which we know BHs may form [14]. The corresponding expression for the Lyapunov coefficient is

$$\lambda r_c = \alpha_c\sqrt{-\frac{1}{6}[11 r_c C'_{\rm rp}(r_c) + r_c^2 C''_{\rm rp}(r_c)]}, \qquad (16)$$

which, for the argument of the previous section, vanishes at the first value of r_c for which Eq. (15) is satisfied.

The logic now is the following. The condition of vanishing Lyapunov coefficient selects a critical radius r_c, while the condition (15) selects the amplitude of the compaction function at that r_c. Larger values of the compaction function will have nonvanishing Lyapunov coefficients and therefore unstable circular orbits.

Let us take as an example the compaction function of the form

$$C_{\rm rp}(r) = A_{\rm rp}(r/r_0)^2 \exp[(1 - (r/r_0)^{2k})/k], \qquad (17)$$

which is the same as in Ref. [18], but in the radial polar gauge. The condition $\lambda = 0$ gives

$$(r_c/r_0)^{2k} = \frac{1}{2}\left(7 + k - \sqrt{25 + 14k + k^2}\right), \qquad (18)$$

which gives $r_c \sim 0.9\ r_0$ for $k \simeq 0$ and $r_c \sim r_0$ for $k \gg 1$. [We choose the − branch of the square root because the other solution as $k \to 0$ gives a vanishing compaction function $C_{\rm rp}(r_c)$.] Imposing the condition (15) fixes the value of $A_{\rm rp}$ and correspondingly of $C_{\rm rp}(r_c)$, which turns out to be $C_{\rm rp}(r_c) \simeq 0.6$ for $k \ll 1$ and $C_{\rm rp}(r_c) \simeq 0.5$ for $k \gg 1$.

Our goal is now to compare the maximum value of the compaction $C_{\rm rp}(r)$ determined in this way with the critical value of the compaction function to form a BH calculated numerically in the comoving gauge and on superhorizon scales and well fitted by the formula [18–20],

$$C^{\rm c}_{\rm com}(\tilde{r}_m) = \frac{4}{15}e^{-1/q}\frac{q^{1-5/2q}}{\Gamma(5/2q) - \Gamma(5/2q, 1/q)}, \qquad (19)$$

where r_m is the location of the maximum of the compaction function and $q = -\tilde{r}_m^2 C''_{\rm com}(\tilde{r}_m)/4C_{\rm com}(\tilde{r}_m)$.

To do so, we have to go from the radial polar gauge to the comoving gauge [21] defined with spatial coordinates $\tilde{r}$ by knowing that the compaction function is coordinate invariant [22]

$$C_{\rm com}(\tilde{r}) = C_{\rm rp}[r(\tilde{r})]. \qquad (20)$$

Explicitly,

$$C^{\rm null}_{\rm rp}(r) \to C^{\rm null}_{\rm com}(r) = 1 - \frac{1 - C^{\rm null}_{\rm rp}(r)}{\alpha(r)}. \qquad (21)$$

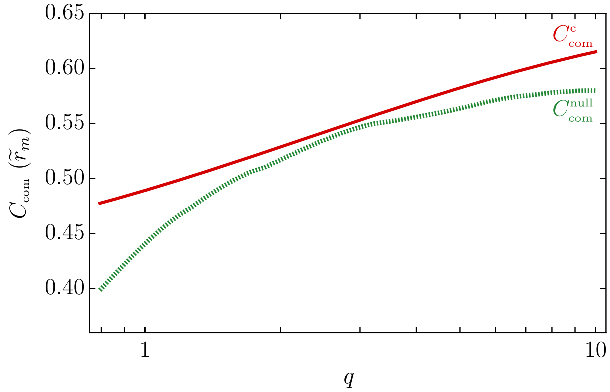

FIG. 3. The comparison between the critical value of the compaction function from Refs. [18–20] (red line) and the one obtained from the existence of the first circular orbit (dotted green line).

By evaluating it at its maximum $\tilde{r}_m(r_m)$ for which we have calculated the corresponding value of q, and by demanding the existence of the critical value r_c such that $\lambda = 0$ in the radial polar gauge, we obtain the critical compaction function in Fig. 3 (see Supplemental Material for more details [23]).

The red solid line indicates the value (19), while the dotted green line indicates the compaction function in the comoving gauge corresponding to the appearance of the first unstable circular orbit. Admittedly, our correspondence fails for values of $q \ll 1$. Luckily, realistic models for BH formation do not have values in this regime. This is an impressive result given our assumption of neglecting the initial radial velocity. We have also checked that the result is stable against changing the parametrization (17). We also notice that the two critical values depart more for $q \gg 1$ as the threshold value from the expression (19) tends to $2/3 \simeq 0.66$, while the one from the circular orbit reasoning increases up to ~ 0.6. This discrepancy is not surprising as more peaked compaction functions are characterized by larger pressure gradients and our approximation is supposed to lose its validity in this regime.

Second evidence: The critical exponent from null self-similar geodesics—The gravitational collapse can be briefly described as follows. During its growth, when the comoving Hubble radius reaches the same size of a given overdensity, if the latter is larger than a critical threshold, a BH will form. It is also the moment when the spacetime metric and the energy momentum tensor quickly approach a self-similar behavior [15] which depends only on the variable

$$z = \frac{r}{(-t)}, \qquad t < 0 \tag{22}$$

and is independent from the time variable

$$\tau = -\ln(-t). \tag{23}$$

At later times, self-similarity is broken, leading eventually to the formation of a BH if the evolution is supercritical, that is if the compaction function at its maximum is larger than a critical value (for a review, see Ref. [24]). The resulting BH mass follows a scaling relation of the type [15,16,25]

$$M_{\rm BH} = \mathcal{O}(1) M_{\rm H} (C_{\rm com} - C^{\rm c}_{\rm com})^{\gamma}, \quad \gamma \simeq (0.35 \div 0.37), \tag{24}$$

accounting for the mass of the BH at formation written in units of the horizon mass $M_{\rm H}$ at the time of horizon reentry. The critical exponent γ is universal reflecting a deep property of the gravitational dynamics. During the self-similar solution the dynamics depends only on the variable z and not on the variable $\tau = -\ln(-t)$. The metric (1) is equivalent to

$$\begin{aligned} {\rm d}s^2 &= -f(z){\rm d}\tau^2 + a^2(z){\rm d}z^2 - 2a^2(z) z {\rm d}z\, {\rm d}\tau + z^2 {\rm d}\Omega^2, \\ f(z) &= \alpha^2(z) - z^2 a^2(z). \end{aligned} \tag{25}$$

We can now repeat the same procedure as before to find the Lyapunov coefficient for the perturbed orbit around the critical "radius" z_c which satisfies the conditions

$$\frac{E^2}{L^2} = \frac{f_c}{z_c^2} \quad \text{and} \quad 2f_c = z_c f'_c, \tag{26}$$

where now the prime indicates the derivative with respect to z. The Lyapunov coefficient reads

$$\lambda = \frac{1}{\sqrt{2}} \sqrt{\frac{f_c}{z_c^2 \alpha_c^2 a_c^2} (2f_c - z_c^2 f''_c)} \tag{27}$$

and determines the timescale of the unstable circular orbits

$$\delta z = \delta z_0 e^{\lambda \tau}. \tag{28}$$

We now note that δz_0 will be proportional to $(C_{\rm com} - C^{\rm c}_{\rm com})$ for a family of geodesics that approach the unstable orbit when $C_{\rm com} = C^{\rm c}_{\rm com}$. Perturbation theory breaks down when $\delta z \simeq 1$, which sets the time when the geodesic will depart from the circular orbit [13]. We find from Eq. (28)

$$(C_{\rm com} - C^{\rm c}_{\rm com})(-t)^{-\lambda} \sim 1. \tag{29}$$

On the other hand, the BH mass $M_{\rm BH}$ inside the apparent horizon is related to its radius by $M_{\rm BH} = r_{\rm H}/2G_N$. Replacing $-t_{\rm H} = r_{\rm H}/z_{\rm H}$, we find

$$M_{\rm BH} \sim (C_{\rm com} - C^{\rm c}_{\rm com})^{1/\lambda}. \tag{30}$$

Now, the value of the Lyapunov coefficient can be extracted running the self-similar simulations following Ref. [16] (and whose results we will report elsewhere [26]) and we

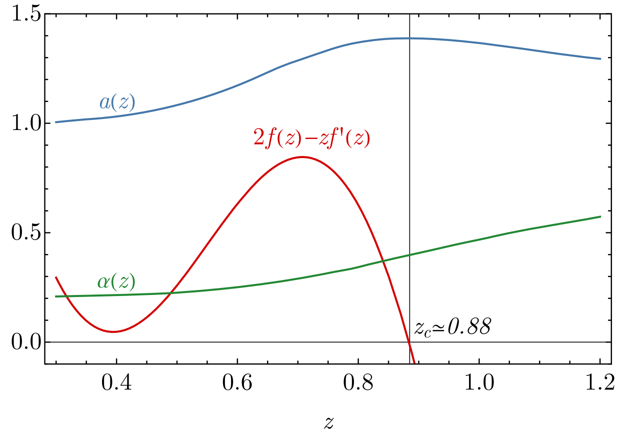

FIG. 4. The values of the functions entering the Lyapunov coefficient in Eq. (27) reproducing the data from Ref. [16]. The chosen time is $\tau = -0.5$, but the behavior is self-similar, valid for all values of τ till self-similarity is broken at the formation of the BH. z_c is determined by the condition $2f(z) - zf'(z) = 0$.

have found $\lambda \simeq 2.9$, giving $\gamma = 1/\lambda \simeq 0.35$, which is very close to the value observed numerically in the literature [15,16,25].

An estimate to understand this result is the following. From Fig. 4, obtained by reproducing the results of Ref. [16], one can appreciate that $z_c \simeq 1$ and $f(z) \simeq -z^2 a^2(z)$. Under these approximations, the Lyapunov coefficient turns out to be $\lambda \sim (z_c^2/\alpha_c)\sqrt{a_c'' a_c}$. One can estimate $a_c \simeq 1/\sqrt{1 - C_{\rm rp}} \simeq \sqrt{2}$ since $C_{\rm rp}$ is always around 0.5. Similarly, $a_c'' \simeq a_c/z_c^2$. Taking $\alpha_c \simeq 1/2$, the Lyapunov coefficient is $\lambda \simeq 2\sqrt{2} \simeq 2.8$. This gives $\gamma = 1/\lambda \simeq 0.36$, which well approximates the numerical value.

Further comments and conclusions—There is one more piece of evidence of the correspondence we have proposed. Consider the moment when the BH has finally formed. Its mass follows the relation (24) with the critical exponent $\gamma \simeq 0.36$. For a BH, using the Schwarzschild metric, one easily finds that the circular orbit—or photon ring—exists at $r_c = 3G_N M_{\rm BH}$. Following the same logic to find the geodesics with $\alpha^2 = a^{-2} = 1\text{–}2GM_{\rm BH}/r$ in the metric, the corresponding Lyapunov coefficient is, in units of the horizon radius $r_{\rm H} = 2G_N M_{\rm BH}$ and independently from the BH mass,

$$\lambda = \sqrt{\frac{f_c}{2r_c^2}(2f_c - r_c^2 f_c'')}, \qquad f_c = 1 - \frac{r_{\rm H}}{r_c}, \tag{31}$$

which corresponds to

$$\lambda r_{\rm H} = \frac{2}{3\sqrt{3}} \simeq 0.38. \tag{32}$$

The approximate equality between this value and the value of the critical exponent γ (not its inverse) is striking. Furthermore, this (maybe only apparent) coincidence resembles the similarity—which we have mentioned in the introduction—between the scaling exponent setting the number of orbits of two Schwarzschild BHs before merging into a Kerr BH and the Lyapunov coefficient of the circular orbit geodesics of the final Kerr BH final state [13]. While we honestly do not have at the moment an understanding of such similarity, it will be interesting to investigate whether it is further evidence of the correspondence between the formation of BHs and null geodesics. Other possible directions are to check if the correspondence works for other equations of state or other matter collapsing fields, e.g., massless scalar fields.

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[1] S. N. Zhang, W. Cui, and W. Chen, Astrophys. J. Lett. **482**, L155 (1997).
[2] R. Narayan, New J. Phys. **7**, 199 (2005).
[3] S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarfs and neutron Stars. The Physics of Compact Objects* (1983).
[4] W. H. Press, Astrophys. J. Lett. **170**, L105 (1971).
[5] H.-P. Nollert, Classical Quantum Gravity **16**, R159 (1999).
[6] K. D. Kokkotas and B. G. Schmidt, Living Rev. Relativity **2**, 2 (1999).
[7] C. J. Goebel, Astrophys. J. **172**, L95 (1972).
[8] V. Ferrari and B. Mashhoon, Phys. Rev. D **30**, 295 (1984).
[9] B. Mashhoon, Phys. Rev. D **31**, 290 (1985).
[10] E. Berti and K. D. Kokkotas, Phys. Rev. D **71**, 124008 (2005).
[11] N. J. Cornish and J. J. Levin, Classical Quantum Gravity **20**, 1649 (2003).
[12] V. Cardoso, A. S. Miranda, E. Berti, H. Witek, and V. T. Zanchin, Phys. Rev. D **79**, 064016 (2009).
[13] F. Pretorius and D. Khurana, Classical Quantum Gravity **24**, S83 (2007).
[14] E. Bagui *et al.* (LISA Cosmology Working Group), arXiv:2310.19857.
[15] M. W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).
[16] C. R. Evans and J. S. Coleman, Phys. Rev. Lett. **72**, 1782 (1994).
[17] I. Musco, J. C. Miller, and L. Rezzolla, Classical Quantum Gravity **22**, 1405 (2005).
[18] A. Escrivà, C. Germani, and R. K. Sheth, Phys. Rev. D **101**, 044022 (2020).

[19] I. Musco, Phys. Rev. D **100**, 123524 (2019).
[20] I. Musco, V. De Luca, G. Franciolini, and A. Riotto, Phys. Rev. D **103**, 063538 (2021).
[21] T. Harada, C.-M. Yoo, T. Nakama, and Y. Koga, Phys. Rev. D **91**, 084057 (2015).
[22] C. W. Misner and D. H. Sharp, Phys. Rev. **136**, B571 (1964).
[23] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.133.081401 for further details on the steps performed to obtain Fig. 3.
[24] C. Gundlach, Phys. Rep. **376**, 339 (2003).
[25] I. Musco, J. C. Miller, and A. G. Polnarev, Classical Quantum Gravity **26**, 235001 (2009).
[26] A. Ianniccari *et al.* (to be published).