Reclaiming the Lost Conformality in a Non-Hermitian Quantum 5-State Potts Model

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Conformal symmetry, emerging at critical points, can be lost when renormalization group fixed points collide. Recently, it was proposed that after collisions, real fixed points transition into the complex plane, becoming complex fixed points described by complex conformal field theories (CFTs). Although this idea is compelling, directly demonstrating such complex conformal fixed points in microscopic models remains highly demanding. Furthermore, these concrete models are instrumental in unraveling the mysteries of complex CFTs and illuminating a variety of intriguing physical problems, including weakly first-order transitions in statistical mechanics and the conformal window of gauge theories. In this work, we have successfully addressed this complex challenge for the $(1+1)$-dimensional quantum 5-state Potts model, whose phase transition has long been known to be weakly first order. By adding an additional non-Hermitian interaction, we successfully identify two conjugate critical points located in the complex parameter space, where the lost conformality is restored in a complex manner. Specifically, we unambiguously demonstrate the radial quantization of the complex CFTs and compute the operator spectrum, as well as new operator product expansion coefficients that were previously unknown.

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Introduction—Conformal field theory (CFT) provides a comprehensive framework for understanding continuous phase transitions and associated critical phenomena [1,2]. However, there are scenarios where conformal symmetry is lost, particularly when conformal fixed points collide and disappear in real physical space [3], leading to first-order transitions. A recent intriguing theory posits that after such collisions, these conformal fixed points relocate to the complex plane [4–6], where conformal symmetry is reestablished in a complex manner. These phenomena give rise to what are now known as complex CFTs [5,6]. Complex CFTs, distinguished by unique properties like complex scaling dimensions, represent a fundamentally new category of nonunitary CFTs compared to well-known examples such as the 2D Lee-Yang singularity in minimal models [2,7,8].

Beyond their theoretical significance, complex CFTs are pivotal in addressing many unresolved problems. They play an essential role in clarifying weakly first-order phase transitions, including the extensively studied deconfined phase transitions [4,5,9–15], and are key in accurately determining the conformal window of critical gauge theories [3,5,16–18]. However, most efforts to address these problems have been concentrated on weakly first-order transitions in the unitary parameter space, where at best, one can observe an approximate conformal symmetry as well as walking (also called pseudocritical) renormalization group (RG) flow influenced by the nearby complex fixed points [15,19,20]. There is a pressing need to harness the full potential of complex conformality, which involves studying the complex fixed points in microscopic models extended to the nonunitary parameter space. Additionally, such microscopic realizations are invaluable for understanding complex CFTs themselves, whose properties remain enigmatic, even in the simplest examples, due to the limitation of existing theoretical tools [5,6,16,18,21–28], where the majority of previous efforts relied on analytical continuation or perturbative RG. A related recent work [27] verified that the lowest two scaling dimensions and complex central charge within two-dimensional classical $O(n>2)$ model also align with the analytical continuation results, while their work did not directly showcase the underlying conformal symmetry.

In this Letter, we successfully tackle the complex challenge for the two-dimensional 5-state Potts model, a model long known for its subtly nuanced weakly first-order transition [29–36]. Specifically, we introduce a $(1+1)$-dimensional non-Hermitian quantum 5-state Potts model and identify its two conjugate complex fixed points. At the complex critical point, employing the state-operator correspondence [37–39], we observe clear evidence of emergent conformal (i.e., Virasoro) invariance and extract the complex scaling dimensions of 11 low-lying Virasoro primary fields [40,41]. Additionally, we calculate 9 distinct operator

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product expansion (OPE) coefficients involving primary fields, most previously unknown. Our results not only directly confirm the complex origin of the weakly first-order phase transition in the two-dimensional 5-state Potts model, but also represent a solid advancement towards unraveling three-dimensional mysteries, such as deconfined phase transitions and critical gauge theories, using the recently proposed fuzzy sphere regularization [15,42].

Model—The Hamiltonian of the Q-state quantum Potts model is [35]

$$H_0(J,h) = -\sum_{i=1}^{L}\sum_{k=1}^{Q-1}\left[J(\sigma_i^\dagger\sigma_{i+1})^k + h\tau_i^k\right], \qquad (1)$$

where the spin shift operator $\hat{\tau}$ and phase operator $\hat{\sigma}$, respectively, changes the local spin degrees of freedom as $\hat{\tau}|n\rangle = |(n+1) \bmod Q\rangle$ and $\hat{\sigma}|n\rangle = e^{2\pi n i/Q}|n\rangle$. The Hamiltonian is invariant under the spin permutation S_Q symmetry, having two phases; (i) an ordered phase at $J > h$ that spontaneously breaks S_Q symmetry, and (ii) a disordered phase at $J < h$ that respects S_Q symmetry. The order-disorder transition occurs at $J = h$, with its precise position determined by the Kramers-Wannier duality. It is well established that the phase transition is continuous for $Q \leq 4$ but is first order for $Q > 4$ [29–34,36]. Notably, for Q just above 4, such as $Q = 5$, the first-order transition is very weak, characterized by a large correlation length and a small energy gap.

Considering the β function for the subleading singlet operator ϵ', which read $-\mathrm{d}g/\mathrm{d}\ln L = a(4-Q) - g^2$ [21]. By tuning g there are indeed two (real) fixed points for $Q < 4$, one is attractive, corresponding to the Q-state Potts CFT, and the other is repulsive, corresponding to the tricritical Q-state Potts CFT [43–46]. At $Q = 4$, these two fixed points merge into a single fixed point, corresponding to an orbifold free boson CFT [31,47]. For $Q > 4$, the two fixed points collide and disappear in the real axis of g, the phase transition becomes first order rather than continuous [29–36,48,49]. Nevertheless, by performing a naive analytical continuation to the complex coupling g, the previously disappeared fixed point will reemerge [see Fig. 1(a)]. It is proposed that these complex fixed points are still conformal, and are responsible for the weakly first-order transition for Q slightly larger than 4 [4,6,22].

In this work, we propose to consider the following interaction term in addition to the original Potts Hamiltonian:

$$H_1(\lambda) = \lambda\sum_{i=1}^{L}\sum_{k_1,k_2=1}^{Q-1}\left[(\tau_i^{k_1}+\tau_{i+1}^{k_1})(\sigma_i^\dagger\sigma_{i+1})^{k_2} + (\sigma_i^\dagger\sigma_{i+1})^{k_1}(\tau_i^{k_2}+\tau_{i+1}^{k_2})\right]. \qquad (2)$$

We let λ be a complex parameter, so the term $H_1(\lambda)$ breaks Hermiticity. Crucially, the term $H_1(\lambda)$ preserves both

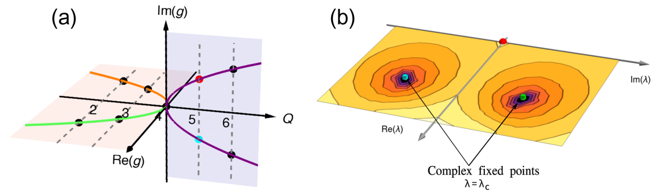

FIG. 1. (a) The complex CFT scenario for the Q-state Potts model: The phase transitions are second order for $Q \leq 4$, while weakly first-order for $Q > 4$ because the location of the critical and tricritical branches merge and move into the complex parameter plane, where g is the coupling constant for the subleading singlet operator ϵ'. (b) Illustration of phase transition points of the non-Hermitian 5-state Potts model $H_{\text{NH-Potts}}(J,h,\lambda)$ on the self-dual plane ($J/h = 1$): Two complex fixed points located at $\lambda = \lambda_c \neq 0$ (green and cyan squares), which are determined via conformal perturbation calculation (see main text). The weakly first order transition point of the original 5-state Potts model is marked by the red dot. The ferromagnetic to paramagnetic phase transitions occur on the self-dual plane $J = h$ (either continuous or first order).

permutation symmetry and self-duality. These properties are essential such that $H_1(\lambda)$ respects all symmetry of the subleading CFT singlet operator ϵ' (see discussion below), which is believed to play a pivotal role in the appearance of complex fixed points [6,21]. We also note that similar lattice construction has been studied in the Potts model to realize UV spin chains for tricritical Ising or tricritical 3-state Potts (real) fixed points [50,51].

In the following, we will directly confirm the complex fixed point proposal by considering the non-Hermitian Potts model, $H_{\text{NH-Potts}}(J,h,\lambda) = H_0(J,h) + H_1(\lambda)$ with complex parameter λ. Specifically, we will identify the complex fixed points for $Q = 5$, and study the corresponding complex CFT through the state-operator correspondence.

Within this work, we only focus on the ferromagnetic Potts model, while the critical behavior for the antiferromagnetic case is more complicated and needs to be treated case by case [52–54]. It was pointed out that 2D real S_5 symmetric CFT could potentially be realized in the non-ferromagnetic Potts model [55,56]. However, this is still an open question that needs further investigation [57].

Critical points in the complex parameter space—We begin by analyzing the phase transition point of $H_{\text{NH-Potts}}(J,h,\lambda)$. First, we assume the microscopic model $H_{\text{NH-Potts}}(J,h,\lambda)$ realizes an effective model $H_{\text{CFT}}(g_{\epsilon'}) = H_{\text{CFP}} + g_{\epsilon'}\int \mathrm{d}x \cdot \epsilon'(x) +$ (irrelevant perturbation), where ϵ' is the most relevant S_Q singlet preserving Kramers-Wannier duality and $g_{\epsilon'}$ is the coupling constant. Tuning parameters to hit the critical value (J_c, h_c, λ_c), $H_{\text{NH-Potts}}(J_c,h_c,\lambda_c)$ realizes a complex CFT fixed point H_{CFP} with vanishing small $g_{\epsilon'}$. Accordingly, one can compare the energy spectrum of $H_{\text{NH-Potts}}(J,h,\lambda)$ with those of $H_{\text{CFT}}(g_{\epsilon'})$, and the minimum of $g_{\epsilon'}$ should point to the location of the critical point of $H_{\text{NH-Potts}}$ [58], as shown in Fig. 1(b). Following this process,

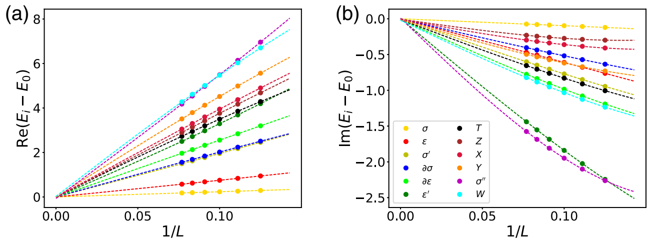

FIG. 2. Finite-size scaling of the low-lying excitation energy gaps (distinguished by colors) at the complex critical point, where (a) real and (b) imaginary part of energy gaps identically scales to zero.

we obtain ($J_c = h_c = 1, \lambda_c = 0.079 + 0.060i$) and its complex conjugated point ($J_c = h_c = 1, \bar{\lambda}_c = 0.079 - 0.060i$), which we determine as the critical points (Supplemental Material Sec. B [59]). Given that the Hermitian breaking term λ_c is very small, the weakly first-order transition point in the original 5-state Potts model (at $J = h = 1$, $\lambda = 0$) is notably close to the complex critical points.

Next, at the critical point, according to the expectation of CFT, the energy spectrum should follow the scaling form

$$(E_n - E_0) = \frac{2\pi v}{L}\Delta_n + \cdots, \tag{3}$$

where L is the length of the spin chain, v is a nonuniversal normalization factor, and E_0 is the ground state energy [37–39]. The "..." stands for finite-size corrections from irrelevant operators (see Supplemental Material Sec. C [59]). It is worth noting that the eigenenergies E_n and scaling dimensions Δ_n of the CFT operator are complex values for complex CFT, in sharp contrast to unitary CFT and nonunitary CFT such as Lee-Yang. As shown in Figs. 2(a) and 2(b), by sitting at the phase transition point the excited energy gaps simultaneously scales to vanishing small values for both the real and imaginary part, signaling the criticality. Additionally, the ground state energy density E_0/L can give rise to the central charge $c \approx 1.1405(2) - 0.0224(2)i$ (for details see Supplemental Material Sec. D [59]), close to the exact result from analytical continuation [6],

$$c_{5\text{-Potts}} = 7 - \frac{12\pi}{2\pi + i\log\left(\frac{3+\sqrt{5}}{2}\right)} - \frac{3i\log\left(\frac{3+\sqrt{5}}{2}\right)}{\pi}$$
$$\approx 1.1375 - 0.0211i.$$

Operator spectrum—Generally, a key feature of the CFT is the state-operator correspondence [37,40], i.e., the eigenstate $|\phi\rangle$ of the CFT quantum Hamiltonians on a cylinder $\mathbb{S}^{d-1} \times \mathbb{R}^1$ has one-to-one correspondence with the CFT operator $\hat{\phi}$, which allows us to access the conformal data such as the scaling dimensions of CFT operators. Previously, the state-operator correspondence has been explicitly shown in CFTs with real scaling dimensions, such as the 2D Ising [40,60,65,66] and nonunitary Lee-Yang CFT [67,68]. Next, we will explicitly demonstrate this correspondence holds in the current model $H_{\text{NH-Potts}}(J_c, h_c, \lambda_c)$, even though it is non-Hermitian.

Figure 3 depicts the obtained operator spectra at the critical point, which are grouped into different conformal families in subfigures. The results have been extrapolated to the thermodynamic limit ($L \to \infty$) and we have rescaled the full spectra by setting energy-momentum tensor to be $\Delta_T = 2$. Here we plot the real-part of the scaling dimensions (for the imaginary part see Ref. [59]). The spectra shows unique and distinguishable features. First of all, in each conformal family, the eigenstate with the lowest real part is identified as the Virasoro primary field, and higher energy states (associated with descendant fields) have nearly integer spacing separating them from the primary field. For example, for a primary operator $\hat{\phi}$ with scaling dimension Δ_ϕ and quantum number (s, s_Q) (here s is the Lorentz spin, and s_Q is spin permutation symmetry quantum number), its descendants can be represented as $L_{-\mu_1}\cdots L_{-\mu_m}\bar{L}_{-\nu_1}\cdots\bar{L}_{-\nu_n}\hat{\phi}$ ($1 \le \mu_1 \le \cdots \le \mu_m$, $1 \le \nu_1 \le \cdots \le \nu_n$), with scaling dimension ($\Delta_\phi + \sum_{i=1}^m \mu_i + \sum_{j=1}^n \nu_j$) and quantum number ($s + \sum_{i=1}^m \mu_i - \sum_{j=1}^n \nu_j, s_Q$) [59]. Importantly, the degeneracy of the descendants satisfy with the expectation of the corresponding Verma module. Note that applying Virasoro generators to some specific states might lead to null states, e.g., the second order descendent of $|\epsilon\rangle$ and the third order descendent of $|\epsilon'\rangle$ [59]. All of the above features demonstrate the emergent conformal symmetry regarding the identified critical point of $H_{\text{NH-Potts}}(J_c, h_c, \lambda_c)$. As far as we know, this hidden complex conformality has not been demonstrated by any quantum Hamiltonian before. Moreover, comparing with the unitary 5-state Potts model (see Fig. S3 [59]), the spectra at the weakly first-order transition point deviates from the CFT tower structure, implying that the transition point there is not exactly continuous.

Having clarified the emergent conformal symmetry at the complex fixed point, we further investigate the scaling dimensions of the identified primary operators, as listed in Table I. Crucially, we find ϵ' ($\Delta_{\epsilon'} \approx 1.908 - 0.599i$), which controls the RG flow around the complex fixed points [6]. Moreover, in comparison with the CFT data from analytical continuation [6,69], the quantitative agreement has been confirmed, i.e., for all 11 Virasoro primary operators that we have identified, the discrepancy is less than 1% for the real and 2% for the imaginary part. Even more remarkably, we have checked all low-lying states with $\mathrm{Re}\Delta_n < 5$, $s \le 3$, and they perfectly match the CFT spectrum (for both primary and descendant operators), without any extra or missing operator (see Tables S2–S11 [59]).

Correlation functions and OPE coefficients—Next we turn to access the OPE coefficients. In general, any local lattice operator O can be expanded by the linear

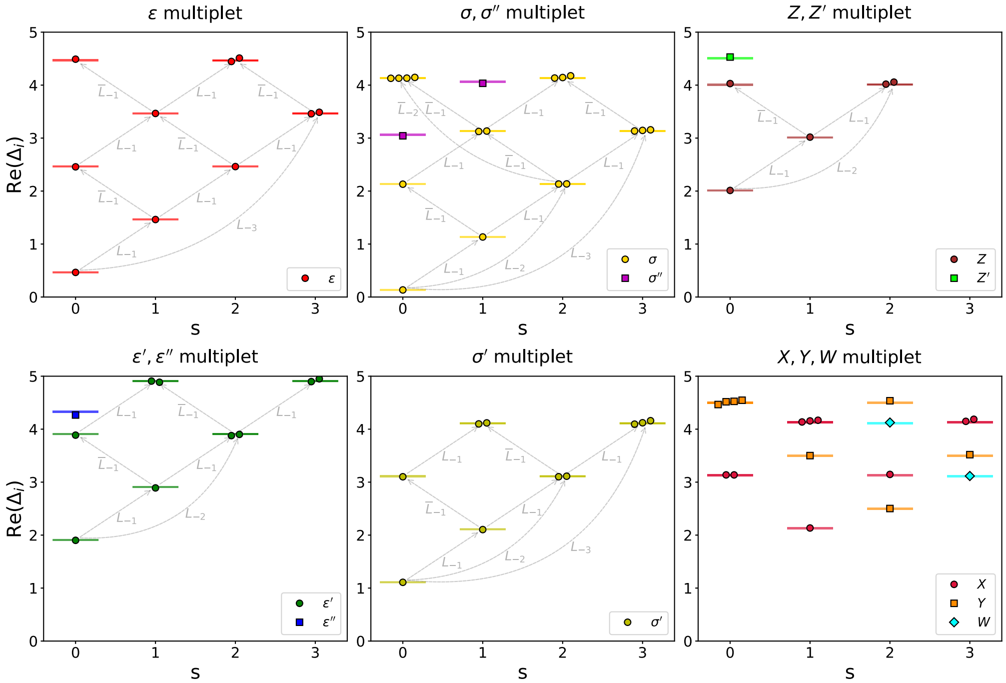

FIG. 3. Conformal multiplet for 11 low-lying Virasoro primary operators: real part of scaling dimension Re(Δ) versus Lorentz spin s. Different symbols and colors label different conformal towers. The spectrum is calibrated by setting the scaling dimension of energy momentum tensor $\Delta_T = 2$. The dots are results from the 5-state Potts model and short lines are predictions from the analytical continuation of the Coulomb Gas partition function [6]. The translucent arrows denote Virasoro generators connecting different states.

combination of CFT operators: $O = \sum_\alpha c_\alpha \hat{\phi}_\alpha$, where the summation is over infinite primary and descendant operators and c_α are some nonuniversal coefficients. For example, spin operator σ_i should involve operator content of CFT operators with the same S_Q quantum number:

TABLE I. Operator scaling dimensions for 11 low-lying Virasoro primary fields identified through the state-operator correspondence. s represents the Lorentz spin quantum number. S_Q Rep is the young diagrams for irreducible representations of the permutation group. The numerical extrapolated data from the non-Hermitian 5-state Potts model is compared with the prediction based on analytical continuation [6]. See the error analysis in Supplemental Material Sec. H [59].

Operator	s	S_Q Rep	Complex CFT	Non-Hermitian 5-Potts
ϵ	0	[5]	$0.466 - 0.225i$	$0.463(12) - 0.224(6)i$
ϵ'	0	[5]	$1.908 - 0.599i$	$1.900(71) - 0.598(15)i$
ϵ''	0	[5]	$4.328 - 1.123i$	$4.340(187) - 1.135(167)i$
σ	0	[4,1]	$0.134 - 0.021i$	$0.133(3) - 0.021(1)i$
σ'	0	[4,1]	$1.111 - 0.170i$	$1.107(29) - 0.171(3)i$
σ''	0	[4,1]	$3.065 - 0.470i$	$3.065(1) - 0.470(1)i$
Z	0	[3,2]	$2.012 + 0.305i$	$2.017(32) + 0.304(19)i$
Z'	0	[4,1]	$4.512 + 0.688i$	$4.536(12) + 0.698(85)i$
X	1	[3,1,1]	$2.134 + 0.286i$	$2.139(26) + 0.285(16)i$
Y	2	[3,2]	$2.500 + 0.230i$	$2.503(13) + 0.230(9)i$
W	3	[3,1,1]	$3.111 + 0.136i$	$3.108(6) + 0.137(5)i$

$$O_\sigma = \sigma_i = (a_\sigma \cdot \hat{\sigma} + \text{desc}) + (b_{\sigma'} \cdot \hat{\sigma}' + \text{desc}) + \cdots, \quad (4)$$

where $\cdots$ represents other CFT operators with higher scaling dimensions. This operator content decomposition can be further inspected by corresponding two-point correlators [59].

Then we calculate correlation functions to extract corresponding OPE coefficients with the help of state-operator correspondence [61]. For example, the OPE $C_{\alpha\sigma\beta}$ can be computed from (Supplemental Material Sec. E [59])

$$\sqrt{\frac{{}_L\langle\alpha|O_\sigma|\beta\rangle_{RL}\langle\beta|O_\sigma|\alpha\rangle_R}{{}_L\langle 0|O_\sigma|\sigma\rangle_{RL}\langle\sigma|O_\sigma|0\rangle_R}} \approx C_{\alpha\sigma\beta} + \frac{c_1}{L^{\Delta_{\sigma'}-\Delta_\sigma}} + \frac{c_2}{L^{2(\Delta_{\sigma'}-\Delta_\sigma)}} + \frac{c_3}{L^2} + O\left(\frac{1}{L^{\Delta_{\sigma'}-\Delta_\sigma+2}}\right), \quad (5)$$

where subscripts L and R represents left and right biorthogonal eigenstates of $H_{\text{NH-Potts}}(J_c, h_c, \lambda_c)$ and $c_{1,2,3}$ are nonuniversal coefficients. The proposed form Eq. (5) is to remove the gauge redundancy of left and right eigenstates. Subsequently a finite-size extrapolation is performed to eliminate the contribution from higher primary and descendant fields. Similarly, other OPE coefficients involving $\epsilon(\epsilon')$ can be obtained by using duality-odd (even) operator $O_\epsilon(O_{\epsilon'})$ [59]. Table II summarizes the obtained 9

TABLE II. OPE coefficients from the non-Hermitian 5-state Potts model, in comparison with analytical continuation from minimal models [6,62]. Some recent work on three/four point correlators in Potts model could be found in [72–76].

OPE	Non-Hermitian 5-Potts	Analytical continuation
$C_{\epsilon\epsilon\epsilon'}$	$0.8781(72) - 0.1433(21)i$	$0.8791 - 0.1404i$
$C_{\epsilon'\epsilon'\epsilon'}$	$2.2591(77) - 1.1916(39)i$	$2.2687 - 1.1967i$
$C_{\epsilon\epsilon'\epsilon''}$	$0.804(3) - 0.220(11)i$	$0.8318 - 0.2027i$
$C_{\epsilon''\epsilon''\epsilon'}$	$3.886(96) - 3.369(33)i$	$3.9261 - 3.3261i$
$C_{\sigma\sigma\epsilon'}$	$0.0658(15) + 0.0513(10)i$	$0.0659 + 0.0527i$
$C_{\sigma\sigma\epsilon}$	$0.7170(6) + 0.1558(1)i$	$0.7154 + 0.1553i$
$C_{\sigma\sigma'\epsilon}$	$0.7520(15) - 0.0831(6)i$	$0.7532 - 0.0811i$
$C_{\sigma\sigma'\epsilon'}$	$0.6062(4) + 0.1664(10)i$	$0.6012 + 0.1630i$
$C_{\sigma\sigma''\epsilon'}$	$0.6709(45) - 0.1250(21)i$	$0.6710 - 0.1143i$

different OPE coefficients. One typical feature distinguished from unitary CFTs is the OPE coefficients are complex values. Recently, the irreducible representation of Virasoro algebra for operators within the ϵ and σ sector of the Potts model for generic Q have been determined to correspond to the Verma module $V_{h_{1,k}}$ and $V_{h_{0,\pm(k-1/2)}}$, where k is a positive integer [70,71]. Since the existence of null states within the ϵ sector and the crossing equation impose significant constraints on related correlators, one could expect to evaluate some OPE coefficients from analytical continuation of previous results of minimal models [62]. Table II shows quantitative agreement between theoretical calculation with numerical results.

Summary and discussion—We have constructed a non-Hermitian 5-state quantum Potts model, which realizes complex fixed points described by an S_5 symmetric complex CFT. At the critical point, we provide compelling evidence for emergent conformal symmetry and present a clear demonstration of state-operator correspondence in complex CFT. Specifically, we have identified the conformal data, including the scaling dimensions of 11 Virasoro primary operators and 9 associated OPE coefficients with high accuracy. These findings are significant in several respects. First, our microscopic calculations shed light on the existence of complex CFT and its corresponding complex criticality. Second, they directly verify the proposal that the first-order regime ($Q > 4$) of the two-dimensional Potts model is in proximity to complex fixed points, paving the way for future investigations into similar models, including gauge theories below the conformal window and deconfined quantum critical points. Third, most of the conformal data computed in this work, such as OPE coefficients, have not been obtained nonperturbatively before. This essential information should facilitate other methods for solving complex conformal data [71,74,76–78]. Finally, this work should also inspire further exploration into quantum critical phenomena in non-Hermitian many-body systems.

Note added—Recently, we become aware of an independent study [79] on the complex fixed points of classical 5-state Potts model.

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