Random Pure Gaussian States and Hawking Radiation

Erik Aurell,[1] Lucas Hackl,[2,3] Paweł Horodecki,[4,5] Robert H. Jonsson,[6] and Mario Kieburg[2]

[1]*KTH—Royal Institute of Technology, Alba Nova University Center, SE-106 91 Stockholm, Sweden*
[2]*School of Mathematics and Statistics, The University of Melbourne, Parkville, Victoria 3010, Australia*
[3]*School of Physics, The University of Melbourne, Parkville, Victoria 3010, Australia*
[4]*International Centre for Theory of Quantum Technologies, University of Gdańsk, Wita Stwosza 63, 80-308 Gdańsk, Poland*
[5]*Faculty of Applied Physics and Mathematics, National Quantum Information Centre, Gdańsk, University of Technology, Gabriela Narutowicza 11/12, 80-233 Gdańsk, Poland*
[6]*Nordita, Stockholm University and KTH Royal Institute of Technology, Hannes Alfvéns väg 12, SE-106 91 Stockholm, Sweden*

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A black hole evaporates by Hawking radiation. Each mode of that radiation is thermal. If the total state is nevertheless to be pure, modes must be entangled. Estimating the minimum size of this entanglement has been an important outstanding issue. We develop a new theory of constrained random symplectic transformations, based on the assumptions that the total state is pure and Gaussian with given marginals. In the random constrained symplectic model we then compute the distribution of mode-mode correlations, from which we bound mode-mode entanglement. Modes of frequency much larger than $[k_B T_H(t)/\hbar]$ are not populated at time t and drop out of the analysis. Among other relatively thinly populated modes (early-time high-frequency modes and/or late modes of any frequency), we find correlations and hence entanglement to be strongly suppressed. Relatively highly populated modes (early-time low-frequency modes) can, on the other hand, be strongly correlated, but a detailed analysis reveals that they are nevertheless very unlikely to be entangled. Our analysis hence establishes that restoring unitarity after a complete evaporation of a black hole does not require any significant quantum entanglement between any pair of Hawking modes. Our analysis further gives exact general expressions for the distribution of mode-mode correlations in random, pure, Gaussian states with given marginals, which may have applications beyond black hole physics.

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Hawking's black hole information paradox has for almost half a century held the place of one of the most important unsolved problems in fundamental physics [1]. The paradox is based on two premises. The first is Hawking's own discovery that black holes radiate [2,3] in the presence of quantum fields. Hawking showed that the state of one mode of frequency about ω localized around some time slice is thermal at the Hawking temperature, which is determined by the black hole mass.

The second premise is that quantum mechanics holds everywhere. Accordingly, closed quantum systems must evolve unitarily if the probabilistic interpretation of quantum theory is to hold true. This implies that an initially pure state will remain pure forever. One can imagine a pure state of matter collapsing into a black hole and then radiating away in Hawking radiation. The total state of all the radiation, after the entire black hole has dissolved and given back its content to the remaining universe, should thus be pure once again. The question is hence how this can be possible if the states of all the single modes are thermal. A selected set of recent reviews discussing the paradox and ways to resolve it are [4,5] and [6]. Page was the first to point out that if the second premise holds, then a resolution of the paradox requires that modes be entangled [7,8].

We recall that the *marginal* over a subsystem A of a large pure state $|\Psi\rangle\langle\Psi|$ is the density matrix $\rho_A = \mathrm{Tr}_B|\Psi\rangle\langle\Psi|$, where B is the complement of A [9]. A resolution of Hawking's paradox following Page is therefore an instance of the quantum marginal problem [10,11]: Given a fixed Hilbert space and a set of disjoint subsystems $\{A_i\}_{i=1}^L$ with complements $\{B_i\}_{i=1}^L$ and density matrices $\{\rho_i\}_{i=1}^L$, does there exist a global pure state $|\Psi\rangle\langle\Psi|$ in $\mathcal{H}$, such that $\rho_i = \mathrm{Tr}_{B_i}|\Psi\rangle\langle\Psi|$ for all $i = 1, \ldots, L$? In general, this is a too involved problem, on the current level of quantum information science. However, if we additionally assume that the total state $|\Psi\rangle\langle\Psi|$ is a pure Gaussian quantum state, the problem is greatly simplified. In [12], two of the present authors showed that the constraints of the Gaussian quantum marginal problem for bosons [13] are easily satisfied for a macroscopic black hole.

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In this work, we show that given any two modes, their expected entanglement is vanishingly small when averaging over all pure Gaussian states whose marginals match Hawking's calculation. To this end, we introduce a new method to the study of random Gaussian states: we consider the family of N-mode pure Gaussian states whose marginals are thermal states. This set of states can be shown to be compact and equipped with a natural measure induced from the Haar measure of the symplectic group. This enables us to find the probability distribution over correlations between two modes at asymptotic infinity $\mathcal{I}^+$. Most of these correlations are very small, which implies that the corresponding mode-mode entanglement is also very small. Some specific pairs of modes (to be described below) can be strongly correlated in the constrained random Gaussian pure state ensemble, but are nevertheless also very unlikely to be entangled.

Relation to other approaches—The literature motivated by Hawking's information paradox is large and still growing. After the publication of the general reviews [4–6], substantial advances have been made: based on ideas from holographic models and AdS [14,15], they led to a partial understanding of microscopic origin of Bekenstein-Hawking entropy through the ideas of entanglement wedges and islands [16–18]. And recently, a family of black hole microstates were proposed as explanation for the microscopic origin of astrophyscial black hole entropy [19,20]. Our approach is complementary to these developments, as we focus on properties of random states, and consider them as a model for Hawking radiation which has escaped the black hole. This point of view was initiated by Page, who showed that the density matrix of a small subsystem is almost surely close to thermal, if the possible pure states of the entire system are sampled uniformly [21].

Page's approach has been applied by substituting the unknown dynamics inside a black hole with a random unitary transformation, and then estimating entanglement between Hawking radiation emitted before and after this internal mixing [21–23]. Compared to Page's approach, and those derived from it, we consider not only one subsystem but a large number of subsystems consisting of all the modes of the system, and we assume that their marginals are all thermal, as they would be in Hawking's theory. Furthermore, motivated by the observation that most modes of the Hawking radiation emanate in a region where gravity is not strong at the black hole horizon, we here consider Gaussian states as approximation to the physical state of the Hawking radiation.

More realistic microscopic models of black hole radiation, however, may yield non-Gaussian states exhibiting subleading corrections. If we had access to a zoo of self-consistent models (classical collapsing black hole spacetime equipped with quantum field, such that Einstein's equations are satisfied with stress energy tensor of the quantum field) to compute the final state, we could analyze how close the respective final states are to the "typical" one we studied. As the few existing models come with their own assumptions (lower dimensions, violations of Einstein's equation, truncation of quantum degrees of freedom, etc.), such a comparison is not available for now. Let us therefore emphasize that our claim is not that we found the correct final state of an evaporating black hole, but that we demonstrated that potential resolutions of the black hole information paradox may require relatively little entanglement between the individual modes.

Pure Gaussian states with fixed marginals—Let $\hat{x}^a = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)$ be quadratures (generalized canonical positions and momenta) of a system with N bosonic degrees of freedom. We here focus on states with vanishing expectation values $\langle \hat{x}^a \rangle = 0$. The covariance matrix of such a quantum state contains the (symmetrized) expectation values,

$$C^{ab} = \frac{1}{2} \langle \hat{x}^a \hat{x}^b + \hat{x}^b \hat{x}^a \rangle. \tag{1}$$

C is a positive semidefinite real symmetric matrix. Its quantum origin is reflected in the Robertson-Schrödinger uncertainty relations that $C + i\Omega \geq 0$ is positive semidefinite as well, where $i\Omega^{ab}\hat{\mathbb{1}} = [\hat{x}^a, \hat{x}^b] = \hat{x}^a\hat{x}^b - \hat{x}^b\hat{x}^a$ is the symplectic form.

Gaussian bosonic states (mixed or pure) are fully characterized by their covariance matrix, which can be decomposed as

$$C = SDS^T \tag{2}$$

where $S \in \mathrm{Sp}(2N)$ is a real symplectic transformation and $D = \mathrm{diag}(d_1, d_1, \ldots, d_N, d_N)$ gives the symplectic eigenvalues. For a pure state, we have $d_i = 1$, i.e., $D = \mathbb{1}$, which means every pure Gaussian state is the vacuum with respect to its normal modes.

The marginal state over one pair of quadrature variables (q_i, p_i) is Gaussian. In general, this state is mixed. It is characterized by the 2×2 covariance matrix $C_{(i)}$ which by construction is real, positive, and positive semidefinite, and also satisfies the Robertson-Schrödinger uncertainty relations. It can therefore be

$$C_{(i)} = S_2 \mathrm{diag}(\mu_i, \mu_i) S_2^T, \tag{3}$$

where $\mu_i \geq 1$ is its symplectic eigenvalue and $S_2 \in \mathrm{Sp}(2)$ is a two-dimensional real symplectic transformation.

Important examples of Gaussian states, the covariance matrices of which are proportional to the identity, are thermal equilibrium states of a single harmonic oscillator mode. For a mode of frequency ω_i and temperature T_i, their covariances are diagonal:

$$C_{(i)} = \mathrm{diag}[\mu_i, \mu_i], \quad \text{with} \quad \mu_i = \coth\left(\frac{\hbar\omega_i}{2k_B T_i}\right). \tag{4}$$

Here, $\frac{1}{2}(\mu_i - 1)$ is the expected number of quanta in the mode, which vanishes only in the vacuum state when the temperature T_i is zero.

In the following, we consider the set $\mathcal{M}_{\hat{C}}$ of all pure Gaussian N-mode state covariance matrices C whose marginals are identical to some prescribed single-mode marginals $C_{(i)}$, collectively referred to as $\hat{C} = \mathrm{diag}(C_{(1)}, \ldots, C_{(N)})$. This set can be equipped with a natural choice of measure $\rho(C|\hat{C})$: as detailed in [24], we obtain it from the restriction of the Haar measure on all symplectic transformations to the compact subset of transformations which [via (2)] generate a covariance matrix C with the prescribed marginals $\hat{C}$. The resulting distribution $\rho(C|\hat{C})$ thus allows us to study the typical two-mode correlations in the ensemble $\mathcal{M}_{\hat{C}}$.

Two-mode correlations and entanglement—To study correlations between two modes of a given N-mode Gaussian state, we consider its reduction to two modes which we call 1 and 2. All the other modes are hence numbered $3, 4, \ldots, N$. The correlation within and between modes 1 and 2 is described by their covariance matrix,

$$C_{1\wedge 2} = \left(\begin{array}{c|c} C_{(1)} & C_{(12)} \\ \hline C_{(12)}^{\intercal} & C_{(2)} \end{array}\right). \tag{5}$$

where $C_{(1)}$ and $C_{(2)}$ are the covariance matrices of modes 1 and 2 as introduced above, and $C_{(12)}$ captures the correlations across variables in mode 1 and mode 2. In the total (large) covariance matrix C, $C_{(1)}$ and $C_{(2)}$ are diagonal 2-by-2 blocks, and $C_{(12)}$ is an off-diagonal 2-by-2 block.

It is well established [25] that one may first separately diagonalize $C_{(1)}$ and $C_{(2)}$ using two symplectic transformations, to get $C_{(i)} = \mu_i \mathbb{1}_2$ $(i = 1, 2)$, and then use special orthogonal transformations (which leave the $\mu_i \mathbb{1}_2$-form of on $C_{(i)}$ unchanged) to get

$$C_{(12)} = C_{(21)} = \mathrm{diag}[c_+, c_-], \tag{6}$$

with $c_+ \geq |c_-|$. The 4-by-4 matrix $C_{1\wedge 2}$ in (5) can therefore without restriction be taken to consist of four diagonal 2-by-2 submatrices. For convenience we shall use the parametrization $d_\pm = (1/\sqrt{2})(c_+ \pm c_-)$, where both $d_\pm$ are non-negative.

The matrix $C_{1\wedge 2}$ in (5) can be a covariance matrix of a quantum state if it satisfies the Robertson-Schrödinger uncertainty relations, which is the case if both its symplectic eigenvalues ν_+, ν_- [being functions of μ_1, μ_2, d_+, and d_-; see [24], Sec. II, Eq. (8)] are larger or equal to one. As shown in Fig. 1, valid Gaussian states lie below a (μ_1- and μ_2-dependent) curve in the $d_\pm$ plane.

A two-mode Gaussian state is entangled if and only if the Peres-Horodecki criterion is satisfied [25]. This criterion is generally given in terms of positivity after a partial transpose, which for two-mode Gaussian states reduces to the statement that smallest symplectic eigenvalue (ν_-^{PT}) of a partially transposed density matrix is less than one. The dashed red line in Fig. 1 is given by $\nu_-^{PT} = 1$, and hence separates entangled from not entangled (separable) states. The symplectic eigenvalues $\nu_\pm^{PT}$ are the same algebraic functions as $\nu_\pm$, with only the roles of d_+ and d_- interchanged. To illustrate the above, in Fig. 1 we plot the logarithmic negativity $\mathcal{N} = \max[0, -\ln \nu_-^{PT}]$ of the state in color coding. We see that entangled states only occur in a relatively small corner toward the maximum value of d_-. For large values of $\mu = \mu_1 \approx \mu_2$ detailed estimates show that the size of the area corresponding to entangled states relative to the size of all allowed states falls off at least as $\log(\mu)/\mu^2$; for illustration, see Fig. 3(c).

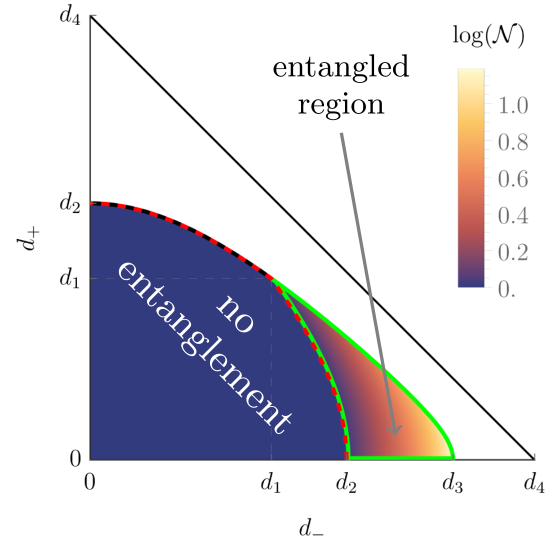

FIG. 1. Correlations and entanglement between two jointly Gaussian modes. Colored region shows the allowed region in the variables (d_+, d_-) parametrizing the mode-mode correlation matrix block $C_{(12)}$, where in this example the two diagonal blocks have symplectic eigenvalues $\mu_1 = 3$ and $\mu_2 = 2$ (see main text). Blue region indicates no entanglement (separable states). Region bounded by green curve indicates entangled states, colored from blue to yellow by increasing logarithmic negativity. Maximal entanglement is found at the tip at $(d_+, d_-) = (0, d_3)$. Marked points in the figure are $d_1 = \sqrt{(\mu_1^2 - 1)(\mu_2^2 - 1)/(2\mu_1\mu_2)}$, $d_2 = \sqrt{2(\mu_1 - 1)(\mu_2 - 1)}$, and $d_3 = \sqrt{2(\mu_1 + 1)(\mu_2 - 1)}$; see [24].

Induced probability distribution over two-mode correlations—The distribution over the ensemble of all pure Gaussian states with fixed marginals $\hat{C} = \mathrm{diag}(C_{(1)}, \ldots, C_{(N)})$ induces—through the two-mode covariance matrix (5)—a distribution on its 2×2 off-diagonal block $C_{(12)}$ [as before, the pair (12) may stand for any fixed pair of modes]. This probability distribution, denoted by $\rho(C_{(12)}|\hat{C})$, is one of the central quantities in this Letter since it gives the typical behavior of two-mode correlations in a random N-mode Gaussian state with fixed single-mode marginals. We construct it explicitly (see Supplemental Material [24]), showing, in particular, that it may be reinterpreted just as a probability measure $\rho(d_+, d_-|\hat{C})$ over the two real, non-negative numbers $d_+, d_- \geq 0$, already known from above to basis-independently characterize the state's two-mode correlations. In the limit of many marginals $\rho(d_+, d_-|\hat{C})$ takes the form

$$\rho(d_+, d_-|\hat{C}) \propto d_+ d_- \frac{\chi(d_+, d_-) \prod_{i=3}^{N} \Phi(\mu_1, \mu_2, C_{(12)}, \mu_i)}{[(\nu_+^2 - 1)(\nu_-^2 - 1)]^2}, \tag{7}$$

where $\chi(d_+, d_-) = \Theta(\nu_- - 1)\Theta(\sqrt{2\mu_1\mu_2} - d_+ - d_-)$ denotes the characteristic function of the region of valid states. Only states i with μ_i strictly larger than one contribute to the product; the explicit form of Φ together with its properties [33] are derived and discussed in [24].

For a large number of modes, these properties typically lead to a concentration of the probability distribution around $d_+ = d_- = 0$. In this case, the probability density can be approximated by a Gaussian matrix model which is obtained by applying $\exp[\log(\ldots)]$ to $\rho(d_+, d_-|\hat{C})$, and performing a leading order expansion of $\log \Phi$ in d_+ and d_-. This results in

$$\rho(d_+, d_-|\hat{C}) \approx 4\Lambda_+\Lambda_- d_+ d_- e^{-\Lambda_+ d_+^2 - \Lambda_- d_-^2}, \tag{8}$$

where Λ_+ and Λ_- are eigenvalues of the bilinear form describing the Gaussian distribution, which can be written as (for other forms of the sums, see [24])

$$\Lambda_\pm = \sum_{j=3}^{N} \left[\frac{\mu_1\mu_2 \pm 1}{(\mu_1^2 - 1)(\mu_2^2 - 1)} - \frac{\mu_j^4 \mu_1 \mu_2 \pm \mu_j^2}{(\mu_j^2\mu_1^2 - 1)(\mu_j^2\mu_2^2 - 1)} \right]. \tag{9}$$

Correlations in evaporating black hole Hawking radiation—Hawking in [2,3] showed that in a Schwarzschild black hole spacetime formed from gravitational collapse, observers at future null infinity $\mathcal{I}^+$—far outside the black hole and long after its formation—observe a quantum field, which at past infinity was in its vacuum state, to be in a thermal state. That is, that the black hole emits thermal radiation at the Hawking temperature $T_H(M) = m_P^2 c^2/(8\pi k_B M)$.

Hawking's derivation uses wave packet modes,

$$a_{nj} = \frac{1}{\sqrt{\Delta\omega}} \int_{j\Delta\omega}^{(j+1)\Delta\omega} \mathrm{d}\omega \, e^{i2\pi n\omega/\Delta\omega} a_\omega, \tag{10}$$

obtained from the continuum s-wave modes a_ω with positive frequency. These modes are characterized by their bandwidth $\Delta\omega$ which is a free parameter in Hawking's theory. The inverse bandwidth $\Delta t = 2\pi/\Delta\omega$ is the width of the mode in time and increasing $n \to n+1$ shifts the center of the mode by Δt. This construction is given in the original papers [3,34], and for convenience reviewed in [24].

Hawking showed that the expectation value $\mu_{nj} = 2\langle a_{nj}^\dagger a_{nj}\rangle + 1$ for late times (large n) and sufficiently large frequencies (large j) is thermal as in (4). In [34], Wald showed that $\langle a_{nj}^\dagger a_{nj}^\dagger\rangle = \langle a_{nj} a_{nj}\rangle = 0$ so that the covariance matrix of one mode has the form (4). Wald further also showed that for the same time slice (same value of n) two-mode correlations vanish [34]. Page and others considered non-s-wave terms. These have the form (4) with $(\mu_i - 1)$ multiplied by graybody factors. These decrease quickly with l such that the radiated power is dominated by s-wave modes.

For the evaporating black hole we posit that its radiation is as in Hawking's theory with suitable Δt. On the one hand, Δt must be chosen (much) shorter than the remaining lifetime of the black hole, which scales as $(M/m_P)^3$. On the other hand, Δt must be longer than the internal timescale of the black hole which different authors have variously suggested to scale as (M/m_P) [35] or $(M/m_P)^2$ [26]. We here opt for the larger Δt, which we call an *epoch*. Note that epochs cannot be of constant length, but must become shorter as the black hole loses mass. Including numerical constants we define the length of epoch n of a black hole of mass M_n to be the time it takes that black hole to radiate away one Planck mass in Hawking's theory, which means

$$\Delta_n = \kappa t_P (M_n/m_P)^2. \tag{11}$$

One epoch corresponds to a slice of Schwarzschild spacetime as indicated in Fig. 2. The dimensionless constant κ depends on the type and number of fields taken into account, and Page estimated it to be on the order of 10^4 or below [27].

To each epoch, we now associate one infinite family of modes a_{nj} as above, where n indicates the epoch and $j = 0, 1, \ldots$ for each epoch. In particular, we now assume the bandwidths of the modes to depend on n as $\Delta\omega_n = 2\pi/\Delta_n$ so that their width in time corresponds to the epoch's length. We assume that these modes have thermal marginals; that is, they have the form (4) with

$$\mu_{nj} = \coth[(j+1)\eta_n], \qquad \eta_n = \frac{\hbar\Delta\omega_n}{2T_n k_B} = \frac{8\pi^2 m_P}{\kappa M_n}. \tag{12}$$

Using that $\eta_n \ll 1$ remains small, the product in (7) and the sum in (8) can be approximated by integrals [24]. To this end, in (7) we write $\prod_{n,j} \Phi_{nj} = \exp(\sum_{n,j} \log \Phi_{nj})$ and approximate the summation over j by an integral for each epoch n. This integral takes a value F which depends only on μ_1, μ_2, and $d_\pm$ but not on n, which is divided by η_n. Hence the summation over n factorizes out and yields a factor of

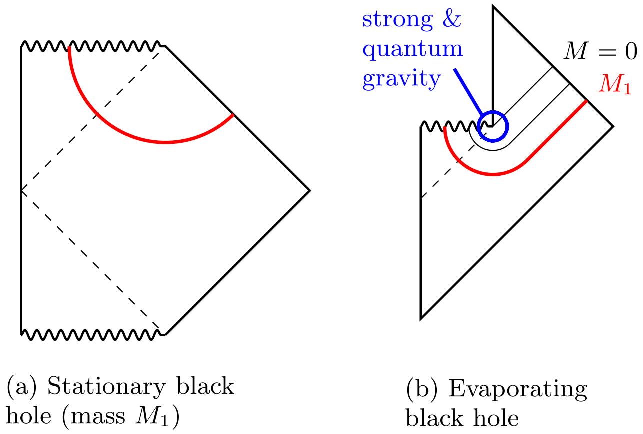

FIG. 2. Penrose diagrams representing stationary and evaporating black holes (following [36]). Dashed lines indicate horizons, wiggly lines singularities, and 45° lines future or past null infinity ($\mathcal{I}^{\pm}$). (a) Diagram of a stationary black hole with fixed mass. Red curve: a time slice of modes escaping to $\mathcal{I}^{+}$. Lower singularity is in the past (white hole), upper singularity in the future (black hole). (b) Diagram of an evaporating black hole obtained by joining time slices of decreasing black hole mass. This spacetime only has one singularity inside the black hole. The red curve corresponds to the same time slice as in (a). The blue ring indicates the region where the curvature near the horizon becomes large enough, such that backreaction and eventually quantum gravitational effects become relevant. Most modes of the outgoing radiation emanate outside this region motivating our assumption of Gaussianity.

$$\sigma = \sum_n \frac{1}{\eta_n} \approx \frac{\kappa}{16\pi^2} \frac{M_i^2 - M_R^2}{m_P^2}, \tag{13}$$

where M_i is the initial mass of the black hole, and M_R is some final mass entering the problem definition, for instance, beyond which Hawking's derivation is no longer valid. For the following M_R^2 can be neglected compared to M_i^2. With this integral approximation the probability distribution reads

$$\rho(d_+, d_-|\hat{C}) = \frac{d_+ d_- [(\nu_+^2 - 1)(\nu_-^2 - 1)]^{-2} \chi(d_+, d_-) e^{\sigma F}}{\prod_{i=1,2} \Phi(\mu_1, \mu_2, C_{(12)}, \mu_i)}, \tag{14}$$

where

$$F = \operatorname{arccoth}^2(\mu_1) + \operatorname{arccoth}^2(\mu_2) - \operatorname{arccoth}^2(\nu_+) - \operatorname{arccoth}^2(\nu_-). \tag{15}$$

Similarly, the integral approximation for the Gaussian approximation (8) reads [24]

$$\Lambda_\pm \approx \sigma \frac{\left((\mu_1^2 - 1)\operatorname{arccoth}(\mu_2) \mp \operatorname{arccoth}(\mu_1)(\mu_2^2 - 1)\right)}{(\mu_1 \mp \mu_2)(\mu_1^2 - 1)(\mu_2^2 - 1)}. \tag{16}$$

For most modes in our ansatz, their symplectic eigenvalue μ_{nj} is not very large and the Gaussian approximation applies. In particular, due to the large numerical prefactor from $\sum_n \eta_n^{-1}$, being proportional to $\kappa \approx 10^4$ and to the square of the initial black hole mass, the probability distribution $\rho(d_+, d_-|\hat{C})$ is narrowly concentrated near $d_\pm = 0$, i.e., at vanishing correlations. In particular, as seen in Fig. 3(a), this means that although for modes with lower μ_{nj} the entangled area corresponds to a sizable portion of the state space, the probability distribution is peaked away from entangled states.

For the most highly excited modes in our ansatz the Gaussian approximation does not apply, but the probability distribution takes the shape seen in Fig. 3(c). The two most highly excited modes are the modes with $j = 0$ from the earliest two epochs, i.e., the modes a_{10} and a_{20}. They have symplectic eigenvalues of $\mu_{10} \approx \kappa M_i/(8\pi^2 m_P)$ and $\mu_{20} \approx \kappa(M_i - m_P)/(8\pi^2 m_P)$. Considering the probability distribution for these two most highly excited modes, it is useful to express the symplectic eigenvalues of their joint two-mode state relative to μ_{10}, i.e., by setting

$$\nu_\pm = \frac{\kappa M_i}{8\pi^2 m_P} y_\pm. \tag{17}$$

By an asymptotic analysis of the exact probability distribution it can then be shown that the probability distribution for large black hole masses converges to

$$\rho(d_+, d_-|\hat{C}) \propto \frac{4\pi^2}{\kappa} \frac{y_+^2 - y_-^2}{y_+^3 y_-^3} e^{-(4\pi^2/\kappa)(y_+^2 + y_-^2)/(y_+^2 y_-^2)} \chi(d_+, d_-). \tag{18}$$

The shape of the distribution relative to μ_{10} is hence strongly dependent on the nondimensional constant κ. This effect can be traced back to our choice of the duration of each epoch, and on the assumption that the marginals of each wave packet mode with these lengths in time are thermal. As seen in Fig. 3(c), already at a value of $\kappa = 10^3$ this distribution peaks at relatively large correlations, and the peak moves farther toward the lower right-hand corner when κ is increased farther. However, only a negligibly small portion of the probability distribution reaches into the region of entangled states. This is due to the size of this region relative to the region of all allowed states decreasing rapidly for large $\mu \approx \mu_1 \approx \mu_2$, as stated above. In fact, one can show that the probability to find an entangled state between the two most excited modes in the radiation of the black hole falls of at least as fast as

$$P_{\text{ent}} = \mathcal{O}\left(\kappa^2 (M_i/m_P)^7 e^{-M_i/(4\pi^2 m_P)}\right). \tag{19}$$

For a derivation of this result and a discussion of other asymptotics, see [24].

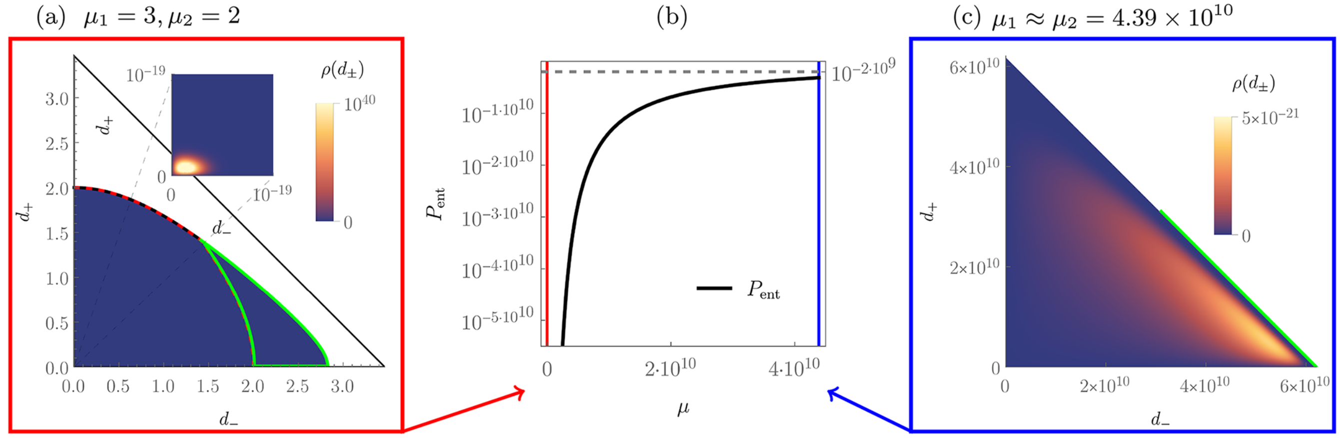

FIG. 3. Allowed mode-mode correlations, entanglement, and probability distributions for Hawking modes. The probability distribution $\rho(d_+, d_-)$ for different values of μ_1 and μ_2 on the background of a black hole spectrum, and its integrated weight over the entangled region. M_i/m_P is 10^{10} and κ is 10^3. (a) $\mu_1 = 3$ and $\mu_2 = 2$. $\rho(d_+, d_-)$ is strongly peaked near the origin (inset). It is well described by the Gaussian approximation [Eq. (8)], and has almost vanishing support in the entangled region (region bounded by green curve). (b) $\mu_1 = \mu_2 = \mu$. P_{ent}, the probability $\rho(d_+, d_-)$ integrated over the entangled region, as function of μ. Dashed horizontal line marks exponential bound [Eq. (19)] on P_{ent} for two modes of Hawking radiation from a black hole. (c) μ_1 and μ_2 are for the two most excited modes from this black hole (corresponding to μ_{10} and μ_{20} in main text), both approximately equal to $\mu = 4.39 \times 10^{10}$. $\rho(d_+, d_-)$ [here obtained from Eq. (7)] is peaked toward the lower right-hand corner. The probability mass of the entangled region (marked in green) is nevertheless small, since it is here a thin slice located beyond the bulk of the probability mass.

Discussion—We have developed the general theory of constrained symplectic transformations, resulting in a mathematically consistent picture of random multimode pure Gaussian states with local modes being thermal. We applied the theory to the Hawking radiation. Considering any fixed two modes in this picture we have shown the vast majority of pure Gaussian states with thermal marginals have zero entanglement between them. They may still admit quantum correlations as characterized by quantum discord [37,38]. The partner mode construction [39–42] guarantees that in a pure Gaussian state to every mode with a mixed state marginal there exists a partner mode such that the two modes are entangled and in a product state with the remainder of the system. Here, our findings suggest that the partner mode for a Hawking mode must be a potentially complicated combination of all other Hawking modes. (In the context of AdS/CFT, see also [43,44].) Quite remarkably, the present result harmonizes very well with a result of Page who calculated that two-mode correlations due to recoil effect in Hawking radiation, though nonzero, are exceedingly small [7]. The present picture may hence suggest that in the search for dynamical mechanisms of black hole evaporation, the assumption of Gaussianity of the resulting radiation may be a good working hypothesis. Independently, the theory of constrained randomized symplectic transformations developed here may be applied in other fields, including in particular the research on thermalization of subsystems of a closed quantum or classical system; see [45–47], and references therein.

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