


Infinite Order Hydrodynamics: An Analytical Example

L. Gavassino *Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240, USA* (Received 11 March 2024; revised 3 May 2024; accepted 27 June 2024; published 17 July 2024)

We construct a kinetic model for matter-radiation interactions whose hydrodynamic gradient expansion can be computed analytically up to infinite order in derivatives, in the fully nonlinear regime, and for arbitrary flows. The frequency dependence of the opacity of matter is chosen to mimic the relaxation time of a self-interacting scalar field. In this way, the transient sector simulates that of a realistic quantum field theory. The gradient series is found to diverge for most flows, in agreement with previous findings. We identify, for the model at hand, the mechanism at the origin of the divergence, and we provide a successful regularization scheme. Additionally, we propose a universal qualitative framework for predicting the breakdown of the gradient expansion of an arbitrary microscopic system undergoing a given flow. This framework correctly recovers all previously known instances of gradient expansion divergence. As a new prediction, we show that the gradient expansion diverges when the energy-dependent mean free path is unbounded above.

DOI: [10.1103/PhysRevLett.133.032302](https://doi.org/10.1103/PhysRevLett.133.032302)

Introduction.—The most pressing open question in relativistic fluid mechanics is: “How far can we push hydrodynamics before it breaks down?” [1–20].

Let us make this question precise. The Knudsen number $\text{Kn} = \lambda/L$ is the ratio between the particles’ mean free path λ (defining the microscopic scale) and the characteristic length scale L of the flow of interest [21,22]. It is common knowledge [23] that, if $\text{Kn} \rightarrow 0$, we can model the system as an ideal fluid, while when $\text{Kn} \gtrsim 1$, we need to rely on microphysics. Thus, there is some value of $\text{Kn} \in (0, 1]$ at which hydrodynamics stops working. Can we identify it precisely?

The answer depends on which hydrodynamic theory one is using. The ideal fluid breaks down at any finite value of Kn , since it is nondissipative, and it predicts that waves survive forever. These issues are fixed in Navier-Stokes theory, whose breakdown is conventionally set at $\text{Kn} = 0.1$ [24]. However, one may try to do even better. Most derivations of hydrodynamics from kinetic theory [25,26] and holography [27,28] express the stress-energy tensor $T^{\mu\nu}$ of a fluid as a formal power series in Kn , known as the “gradient expansion” [24,29],

$$T^{\mu\nu}(\text{Kn}) = T_{\text{ID}}^{\mu\nu} + T_1^{\mu\nu} \text{Kn} + T_2^{\mu\nu} \text{Kn}^2 + \dots, \quad (1)$$

where the zeroth order is the ideal fluid, the first order is Navier-Stokes, the second order is the Burnett equation [30], and higher orders correspond to fluid theories with higher derivative corrections. One hopes that, by adding more and more powers of Kn , the regime of applicability of hydrodynamics will expand more and more, increasing the accuracy of hydrodynamics for all Kn up to the radius of convergence Kn^* of the series (1). Following this line of

thought, it would then be natural to conclude that such radius Kn^* is what ultimately marks the rigorous breakdown of hydrodynamics [31].

Unfortunately, reality turns out to be more complicated. First, there are indications that, in most realistic scenarios, $\text{Kn}^* = 0$ [10,13,32]. This means that, if we keep adding higher and higher orders in Kn , at some point the region of applicability of (1) shrinks instead of expanding. Second, Eq. (1) makes sense only if the function $T^{\mu\nu}(\text{Kn})$ is analytic in Kn . In principle, there may be nonsmooth corrections like $\text{Kn}^{3/2}$ [33], or nonperturbative corrections like $e^{-1/\text{Kn}}$ [34]. Therefore, the breakdown scale of hydrodynamics remains unknown.

The dream of any fluid theorist would be to have a microscopic model where $T^{\mu\nu}(\text{Kn})$ can be computed exactly for arbitrary flow, at arbitrary Kn , in the fully nonlinear regime, and where all the terms $T_n^{\mu\nu}$ in the series (1) have exact analytical formulas. Such model should have realistic interactions, to ensure that the dependence of $T^{\mu\nu}$ on Kn mimics the behavior of some microscopic quantum field theory. Finally, one should be able to extract from it general lessons about the expansion (1). In this Letter, we provide a model that fulfills all these requirements. Note that we work in Minkowski space, with signature $(-, +, +, +)$, and adopt natural units: $c = \hbar = k_B = 1$.

The kinetic model.—Our model setup is a radiation-hydrodynamic system [35–41], namely, a fluid mixture comprised of two substances: a material medium M with negligibly short mean free path (i.e., $\text{Kn}_M \equiv 0$), and a diluted radiation gas R with a finite, possibly large, mean free path (i.e., $\text{Kn}_R > 0$) [42,43]. The former is modeled as an ideal fluid, with a well-defined temperature $T(x^\alpha)$ and

flow velocity $u^\mu(x^\alpha)$. The latter can exist in far-from-equilibrium states, and we must track its kinetic distribution function $f_p(x^\alpha)$ [44,45] (p = momentum). The total stress-energy tensor is the sum of a material nonviscous part and a radiation part:

$$T^{\mu\nu} = T_M^{\mu\nu} + \int \frac{d^3p}{(2\pi)^3 p^0} f_p p^\mu p^\nu. \quad (2)$$

The radiation particles do not interact with each other, but are randomly absorbed and emitted by the medium (neglecting scattering). Thus, Boltzmann's equation is of relaxation type [35,46]. Taking the relaxation time to be $\tau_p = -u_\mu p^\mu/g$ (with $g > 0 = \text{const}$, for simplicity), we have

$$p^\mu \partial_\mu f_p = -u_\mu p^\mu \frac{f_p^{\text{eq}} - f_p}{\tau_p} = g(f_p^{\text{eq}} - f_p). \quad (3)$$

Since τ_p grows linearly with $-u_\mu p^\mu$, our medium is opaque at low frequencies, and transparent at high frequencies [35]. The function f_p^{eq} is the value that f_p would have if the radiation was in local equilibrium, with the temperature T and flow velocity u^μ of the medium. Depending on the statistics, one has $f_p^{\text{eq}} = (e^{-u_\mu p^\mu/T} - b)^{-1}$, $b \in \{0, \pm 1\}$ [47]. The backreaction of the radiation on the motion of the medium follows from the conservation law $\partial_\mu T^{\mu\nu} = 0$ which, combined with (2) and (3), gives

$$\partial_\mu T_M^{\mu\nu} = -g \int \frac{d^3p}{(2\pi)^3 p^0} (f_p^{\text{eq}} - f_p) p^\nu. \quad (4)$$

Why this model?.—The relaxation time $\tau_p \propto$ “energy” mimics the transient behavior of a scalar field theory with ϕ^4 interaction [48]. This choice ensures that the gradient expansion (1) will exhibit “realistic” (i.e., QFT inspired) scaling with powers of $\text{Kn}_R = \max\{\text{Kn}_M, \text{Kn}_R\} = \text{Kn}$. Note that, in QCD plasmas, τ_p scales as a fractional power of the energy [49], while the present model is closer to the kinetics of Yukawa's meson theory [50,51].

Our main reason for relying on radiation hydrodynamics (rather than dealing with ideal gases) is that it allows the introduction of momentum-dependent relaxation times, as in (3), without violating any conservation law [52]. Furthermore, within radiation hydrodynamics, the relaxation-time approximation is *exact* (by Kirchhoff's law [53]), if the medium has negligibly short equilibration timescale [35,36]. By contrast, in ideal gases, equations like (3) are a pretty crude approximation of the Boltzmann collision integral.

Note that the fields T and u^μ have unambiguous meanings in our model, as they characterize the local state of the medium, establishing a natural “hydrodynamic frame” [29]. Indeed, providing initial data for the “matter + radiation” system requires specifying the initial states for both the radiation gas and the medium separately, so that $\{f_p, T, u^\mu\}$ are *independent* dynamical degrees of freedom. This is

convenient, since we do not need to worry about matching conditions [25] (i.e., no integral constraint relates T and u^μ to f_p at a given time).

Linear response properties.—Linearizing Eqs. (2) and (3), and looking for solutions $\propto e^{ik_\mu x^\mu}$, we obtain

$$\delta T^{\mu\nu} = \delta T_M^{\mu\nu} + \int \frac{d^3p}{(2\pi)^3 p^0} \frac{(f_{p,T}^{\text{eq}} \delta T + f_{p,u^\rho}^{\text{eq}} \delta u^\rho) p^\mu p^\nu}{1 + ig^{-1} p^\alpha k_\alpha}. \quad (5)$$

For $k^\alpha = (\omega, 0, 0, 0)$, the integrand is singular whenever $\omega = -ig/p^0$, which approaches $\omega = 0$ for $p^0 \rightarrow \infty$. Thus, the linear-response Green function of the stress-energy tensor has a branch cut that touches the origin of the complex ω -plane. This is a known feature of ϕ^4 [54,55] and other relativistic systems [51], including, probably, QCD plasmas. Hence, the spectral properties of the present model agree with current expectations.

Analytic solution.—Equations like (3) can be solved analytically in the nonlinear regime [35]. Fixed an event x^α and a momentum p^α , consider the worldline

$$x^\alpha(\xi) = x^\alpha - \frac{p^\alpha}{g} \xi, \quad (6)$$

where ξ is the optical depth. Traveling along (6), at fixed momentum p^α , Eq. (3) becomes an ordinary differential equation, $-\dot{f}_p(\xi) + f_p(\xi) = f_p^{\text{eq}}(\xi)$, which has unique solution for initial data at past infinity:

$$f_p(x^\alpha) = \int_0^{+\infty} e^{-\xi} f_p^{\text{eq}} \left(x^\alpha - \frac{p^\alpha}{g} \xi \right) d\xi. \quad (7)$$

Plugging (7) in (2), we obtain an *exact* formula for the total stress-energy tensor:

$$T^{\mu\nu} = T_M^{\mu\nu} + \int \frac{d^3p}{(2\pi)^3 p^0} p^\mu p^\nu \int_0^{+\infty} f_p^{\text{eq}} \left(x^\alpha - \frac{p^\alpha}{g} \xi \right) e^{-\xi} d\xi. \quad (8)$$

This is the function “ $T^{\mu\nu}(\text{Kn})$ ” we were looking for. It remains valid at all orders in derivatives of the fluid fields $\{T, u^\mu\}$, which enter (8) through $f_p^{\text{eq}} = (e^{-u_\mu p^\mu/T} - b)^{-1}$. We may interpret (8) as an “infinite-order hydrodynamic constitutive relation,” where $T^{\mu\nu}$ is a nonlocal functional of T and u^μ , see Fig. 1.

The ideal fluid is recovered in the infinite-opacity limit ($g \rightarrow +\infty$), since $x^\alpha(\xi) \rightarrow x^\alpha$, and the total stress-energy tensor $T^{\mu\nu}$ becomes that of a conglomerate fluid, where matter and radiation comove with velocity u^μ and both have temperature T :

$$\begin{aligned} T_{\text{ID}}^{\mu\nu} &= T_M^{\mu\nu} + \int \frac{d^3p}{(2\pi)^3 p^0} f_p^{\text{eq}} p^\mu p^\nu \\ &= (\varepsilon + P) u^\mu u^\nu + P \eta^{\mu\nu}. \end{aligned} \quad (9)$$

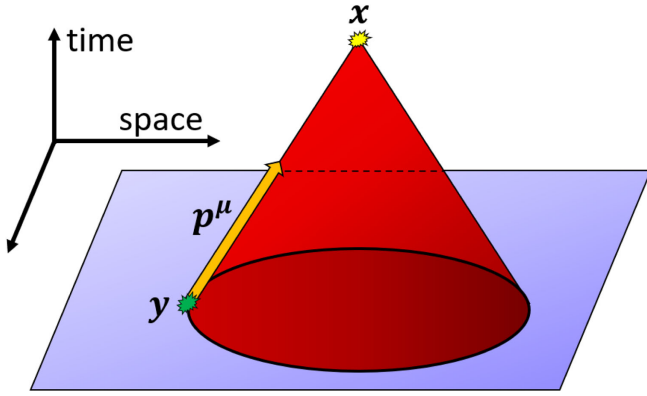


FIG. 1. Visual representation of Eq. (8), viewed as a nonlocal constitutive relation $T^{\mu\nu}[T, u^\alpha]$. The value of $T^{\mu\nu}(x)$ receives contributions from *all* events y on the past light cone of x (although closer events contribute more). The information about $\{T(y), u^\alpha(y)\}$ is transported directly to x by radiation particles emitted at y , which have probability $e^{-\xi} = e^{-g(x^0-y^0)/p^0}$ to reach x before being absorbed.

The energy density $\varepsilon(T, \dots)$ and the pressure $P(T, \dots)$ are the sum of matter and radiation parts in thermal equilibrium, where the matter contributions may depend on additional state variables besides T , including, possibly, some chemical potential.

Gradient series.—Let us evaluate the derivative expansion of (8). If $f_p^{\text{eq}}(x^\mu)$ is an entire function of x^μ , we can express the integrand of (7) as a Taylor series

$$f_p^{\text{eq}}(\xi) = \sum_{n=0}^{+\infty} \frac{\xi^n}{n!} \left(\frac{d}{d\xi} \right)^n f_p^{\text{eq}}(0), \quad (10)$$

where $d/d\xi = -g^{-1}p^\alpha \partial_\alpha$ is the derivative along the parametric line (6). We note that $d/d\xi \sim \text{Kn}$, due to the following chain of estimates (L is the hydrodynamic scale):

$$\frac{d}{d\xi} = -\frac{p^\alpha \partial_\alpha}{g} \sim \frac{T}{gL} \sim \frac{-u^\mu p_\mu}{gL} = \frac{\tau_p}{L} \sim \frac{\lambda}{L} = \text{Kn}. \quad (11)$$

Hence, each n th term in the series (10) corresponds to the power Kn^n in the Knudsen expansion. Plugging (10) into (8), and performing the integral in ξ , we can express the total stress-energy tensor as the sum of the ideal part (9) and a viscous part, namely, $T^{\mu\nu} = T_{\text{ID}}^{\mu\nu} + \Pi^{\mu\nu}$, with

$$\Pi^{\mu\nu} = \sum_{n=1}^{+\infty} (-g)^{-n} \partial_{\alpha_1} \dots \partial_{\alpha_n} \int \frac{d^3 p}{(2\pi)^3 p^0} f_p^{\text{eq}} p^\mu p^\nu p^{\alpha_1} \dots p^{\alpha_n}. \quad (12)$$

This is the gradient series we were looking for. Again, no approximation was made, besides the assumptions of the kinetic model. The integrals in (12) are the moments of the equilibrium distribution function f_p^{eq} , all of which can be evaluated analytically.

Let us discuss some notable properties of (12): (i) All the components of $\Pi^{\mu\nu}$ are in general nonzero, including

$\Pi^{\mu\nu} u_\mu u_\nu$. Thus, the ‘‘material frame,’’ where temperature and flow velocity are those of the medium, does not coincide with the Landau or the Eckart frame, as noted in [43]. (ii) All the contributions to (12) contain factors of the form $\partial_{(\mu_1} \partial_{\mu_2} \dots \partial_{\mu_k} \beta_{\nu)}$, with $\beta_\nu = u_\nu/T$. It follows that $\Pi^{\mu\nu}$ vanishes whenever β_ν is a Killing vector, namely, $\partial_{(\mu} \beta_{\nu)} = 0$, which is the condition of global thermodynamic equilibrium [56–59]. (iii) If we truncate $\Pi^{\mu\nu}$ at finite order $n \geq 1$ (with Maxwell-Boltzmann statistics, i.e., $f_p^{\text{eq}} = e^{\beta_\rho p^\rho}$), the principal part of the conservation law $\partial_\mu T^{\mu\nu} = 0$ (regarded as a dynamical equation for β_ν) is

$$\frac{1}{(-g)^n} \int \frac{f_p^{\text{eq}} d^3 p}{(2\pi)^3 p^0} p^\mu p^\nu p^\rho p^{\alpha_1} \dots p^{\alpha_n} \partial_{\alpha_1} \dots \partial_{\alpha_n} \partial_\mu \beta_\rho. \quad (13)$$

The associated characteristic polynomial [60] is

$$\mathcal{P}(\zeta_\alpha) = \det \left[\int \frac{f_p^{\text{eq}} d^3 p}{(2\pi)^3 p^0} (p^\alpha \zeta_\alpha)^{n+1} p^\nu p^\rho \right]. \quad (14)$$

For n odd, the matrix in the determinant is positive definite for all ζ_α . Thus, the evolution equation is elliptic, making all odd-order truncations acausal, unstable, and ill-posed [61–63].

Divergence of the gradient series.—Let us show that (12) is usually divergent. We can estimate the order of magnitude of each term of the series by making the formal replacements $\partial_\alpha \rightarrow 1/L$ and $f_p^{\text{eq}} \rightarrow e^{-p^0/T}$. Then, Eq. (12) becomes (ignoring constant factors)

$$\frac{\Pi^{\mu\nu}}{T^4} \sim \sum_n \left(\frac{T}{gL} \right)^n (n+3)!, \quad (15)$$

which is factorially divergent.

Let us now repeat the above calculation, assuming that the medium cannot produce radiation particles whose energy exceeds some large cutoff scale $\Lambda \gg T$, thereby restricting integrals in $d^3 p$ to the sphere $p^0 \leq \Lambda$. Then, the series (15) becomes

$$\frac{\Pi^{\mu\nu}}{T^4} \sim \sum_n \left(\frac{T}{gL} \right)^n \gamma(n+4, \Lambda/T), \quad (16)$$

where γ is the lower incomplete Gamma function, which becomes $(n+3)!$ if we send $\Lambda \rightarrow +\infty$ at fixed n . Instead, let us evaluate the series for *finite* $\Lambda \gg T$. At large n , the addends scale like

$$\left(\frac{T}{gL} \right)^n \gamma(n+4, \Lambda/T) \sim \left(\frac{\Lambda}{gL} \right)^n \frac{e^{-\Lambda/T} (\Lambda/T)^4}{n}. \quad (17)$$

If $gL > \Lambda$, the series converges. Furthermore, the large- n part of the series adds up to

$$-e^{-\Lambda/T} \left(\frac{\Lambda}{T} \right)^4 \ln \left(1 - \frac{\Lambda}{gL} \right), \quad (18)$$

which is negligible in the limit $\Lambda \gg T$.

The above argument shows that the divergence of the gradient expansion comes from fictitious particles that carry unreasonably large energy. We can prove that such particles are unphysical by showing that the exact formula (8) is unaffected by the cutoff Λ . Called $T_{\Lambda}^{\mu\nu}$ the regularized stress-energy tensor, we have

$$\begin{aligned} \frac{T^{\mu\nu} - T_{\Lambda}^{\mu\nu}}{T^4} &= \int_{p^0 \geq \Lambda} \frac{d^3 p}{(2\pi)^3 p^0} \frac{p^{\mu} p^{\nu}}{T^4} \int_0^{+\infty} f_p^{\text{eq}}(\xi) e^{-\xi} d\xi \\ &\sim \int_{p^0 \geq \Lambda} \frac{d^3 p}{(2\pi)^3} \frac{p^0 e^{-p^0/T}}{T^4} \sim \left(\frac{\Lambda}{T}\right)^3 e^{-\Lambda/T}, \end{aligned} \quad (19)$$

which tends to zero when $\Lambda \gg T$. This estimate shows that the cutoff Λ can be used as a practical tool, to regularize the gradient expansion (12), and restore agreement with (8). One only needs to make sure that $gL > \Lambda \gg T$. Unfortunately, this also requires $\text{Kn} = T/(gL) \ll 1$. This suggests that adding higher derivative terms may not significantly expand the regime of applicability of hydrodynamics. Instead, it may just lead to a refinement of the accuracy of hydrodynamics, within the same interval $0 \leq \text{Kn} \lesssim 0.1$ where Navier-Stokes is already valid. Indeed, several previous studies seem to support the present conclusion [16,30,64,65].

A quick example.—We can check the consistency of the estimates (18) and (19) with an example. Take $u^{\mu} = (1, 0, 0, 0)$ and $T(t) \propto 1 - a \sin(t/L)$, with $a \rightarrow 0$. Linearizing in a , and defining $\text{Kn} = T/(gL)$, the value of $\Pi^{00}(0)$ according to (8) and (12) is given by, respectively,

$$\text{Exact: } \frac{\Pi^{00}}{aT^4} = \int_0^{+\infty} \frac{dz}{2\pi^2} \frac{\text{Kn} z^5 e^{-z}}{1 + \text{Kn} z^2}, \quad (20)$$

$$\text{Expand: } \frac{\Pi^{00}}{aT^4} = \sum_{n \text{ odd}} \frac{(-1)^{\frac{n-1}{2}}}{2\pi^2} \text{Kn}^n (n+4)!, \quad (21)$$

$$\text{Cutoff: } \frac{\Pi^{00}}{aT^4} = \sum_{n \text{ odd}} \frac{(-1)^{\frac{n-1}{2}}}{2\pi^2} \text{Kn}^n \gamma\left(n+5, \frac{\Lambda}{T}\right), \quad (22)$$

where Eq. (22) is the same as (21), corrected with cutoff $\Lambda/T < \text{Kn}^{-1}$ (for convergence of the series). We observe in Fig. 2 the breakdown of the gradient expansion at order 70 (higher orders explode at smaller Kn). The cutoff Λ restores agreement with (8) at all orders, within an error that survives at large n and increases with Kn . This fixes the breakdown of the regularised gradient series at $\text{Kn} \sim 0.1$, which is precisely when Navier-Stokes ceases to hold [24].

Origin of the divergence.—We now provide a general explanation for the divergence of the gradient expansion, which applies to the present model and to most other models in the literature.

Let A be a density of interest, located at x . By causality, the event x receives influence from all events y in the causal past $J^-(x)$ of x [66]. Information is transported from y to x

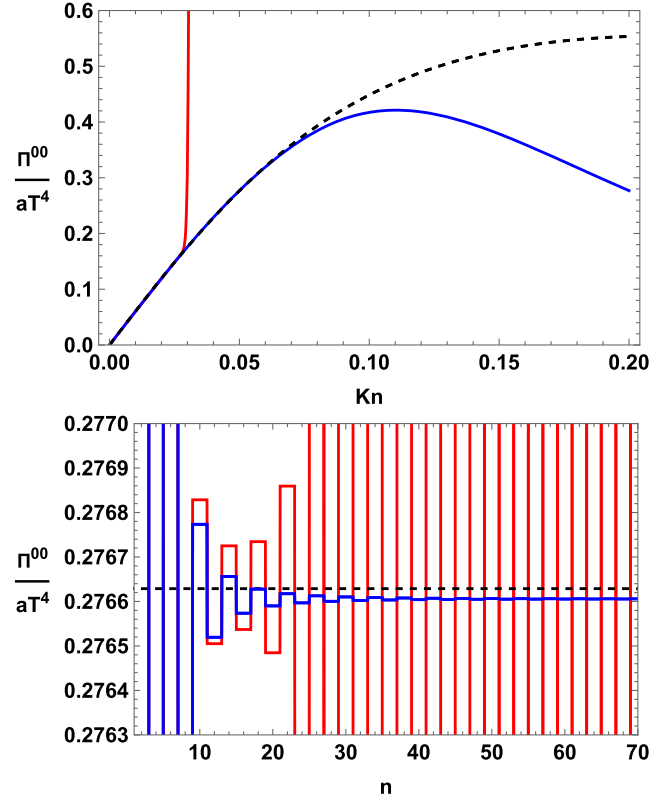


FIG. 2. Upper panel: Exact viscous density $\Pi^{00}(t=0)$ according to (20) (dashed) compared to the nonregularized gradient series (21) (red) truncated at $n = 70$, and the regularized series (22) (blue) with $n \rightarrow \infty$ and cutoff scale $\Lambda/T = (\text{Kn} + 0.01)^{-1}$. Lower panel: Convergence test at $\text{Kn} = 0.05$ (same colors).

by messengers (e.g., particles or excitations), each having an associated energy (or frequency) E . Thermalization erases the messenger’s memory about y over a timescale $\tau(E)$. Thus, we can schematically write

$$A = \int_0^{+\infty} dE \int_{J^-(x)} d^4 y \sigma_A(E, y) e^{-(x^0 - y^0)/\tau(E)}, \quad (23)$$

where $\sigma_A(E, y)$ quantifies how many messengers with energy E are emitted from y , and “how much A ” they carry. The Knudsen expansion (10) corresponds, in this general setting, to a Taylor expansion of $\sigma_A(E, y)$ around x . Since, in the light cone, $|x^j - y^j| \leq x^0 - y^0$, we may write, schematically (recall: $\partial_\alpha \rightarrow 1/L$),

$$\sigma_A(E, y) \sim \sum_n \frac{\sigma_A(E, x)}{n!} \left(\frac{x^0 - y^0}{L}\right)^n. \quad (24)$$

Plugging (24) into (23), and taking the series outside the volume integral, we obtain, up to n -independent factors,

$$A \sim \int_0^{+\infty} dE \sigma_A(E, x) \sum_n \left[\frac{\tau(E)}{L}\right]^{n+1}. \quad (25)$$

Now we see the problem: If $\tau(E)$ is unbounded above, fixed any L , there is some energy E such that $\tau(E) > L$, and the geometric series in (25) diverges for such E . This implies that we were not allowed to exchange the series in n with the integral in d^4y , and, in general, [67]

$$A \left[\sum_{n=0}^{+\infty} \sigma_{A,n} \right] \neq \sum_{n=0}^{+\infty} A[\sigma_{A,n}]. \quad (26)$$

This is precisely the issue with our analytical model. The relaxation time $\tau_p = -u_\mu p^\mu / g$ diverges at high energies, and we cannot take the series (10) outside the integral (8). The introduction of the energy cutoff resolves the problem because, if $\Lambda/T < \text{Kn}^{-1}$, the series (25) converges for all $E \leq \Lambda$, since (see Fig. 1)

$$\frac{\tau(E)}{L} = \frac{E}{gL} \leq \frac{\Lambda}{gL} = \frac{\Lambda}{T} \text{Kn} < 1. \quad (27)$$

The same conclusions apply to all kinetic models of QCD plasmas [49,68,69], whose $\tau(E) \propto E^q$, with $q > 0$ diverges at large E (i.e., for jets). According to (25), the gradient series of all such quark-matter models has a jet-induced divergence, since, fixed L , $\tau(E) \gg L$ when $E \rightarrow \infty$.

With similar reasoning, we can also predict the breakdown of all gradient expansions due to stochastic fluctuations [33]. In fact, long-wavelength hydrodynamic fluctuations can transport information from y to x , and have decay timescale $\tau(E) \sim E^{-2}$ (with $E = \text{frequency}$), which diverges at small E . This generates an infrared (i.e., low E) divergence in (25), consistently with [33].

In the Supplemental Material [70], the above claims are shown to hold (as rigorous statements) within linear-response theory.

General flows.—In the above estimates, we always assumed that $(\partial)^n \rightarrow 1/L^n$. This means that our results apply to sinusoidal and exponential flows, or finite superpositions thereof. However, different flows attribute different weights to each derivative. In Bjorken expansion [77], where everything scales like $(y^0)^{-1}$, the correct formal replacement is $(\partial)^n \rightarrow n!/L^n$, and (25) becomes

$$A \sim \int_0^{+\infty} dE \sigma_A(E, x) \sum_n \left[\frac{\tau(E)}{L} \right]^{n+1} n!, \quad (28)$$

which diverges independently from the value of $\tau(E)$. This explains why Israel-Stewart hydrodynamics has divergent gradient expansion in Bjorken flow [7], despite having only one relaxation time. We also understand why Borel summation gives correct predictions in Bjorken flow [3]. By Borel-summing the series (28), we obtain [78]

$$\sum_{n=0}^{+\infty} \left[\frac{\tau(E)}{L} \right]^{n+1} n! \stackrel{\text{B.P.V.}}{=} \int_0^{+\infty} \frac{e^{-t/\tau(E)}}{L-t} dt, \quad (29)$$

which has precisely the form of the spacetime integral in (23), with $\sigma_A(y)$ that decays like $1/y^0$. This suggests that Borel summation “undoes” the gradient expansion, whose existence was not justified in the first place, and it reconstructs the original (correct) integral (23) from which the gradient expansion was (improperly) derived.

The same reasoning can be generalized to arbitrary flows. For example, suppose that the Taylor expansion of $\sigma_A(y)$ has finite radius of convergence \mathcal{R} . Then, for large n , we have, schematically, $(\partial)^n \rightarrow n!/\mathcal{R}^n$ [79], and

$$A \sim \int_0^{+\infty} dE \sigma_A(E, x) \sum_n \left[\frac{\tau(E)}{\mathcal{R}} \right]^{n+1} n!. \quad (30)$$

This explains why, in most realistic flows, where $T(x^\alpha)$ and $u^\mu(x^\alpha)$ are not entire functions, the gradient expansion is factorially divergent [32]. The general validity of this argument within linear-response theory is demonstrated in the Supplemental Material [70].

Conclusions.—Our analysis reveals what infinite-order hydrodynamics looks like. The stress-energy tensor is still a functional of the fluid fields $\{T, u^\alpha\}$, just like in finite-order hydrodynamics. However, such a functional $T^{\mu\nu}[T, u^\alpha]$ is *non-local* at infinite order [80]. By causality, $T^{\mu\nu}(x)$ depends solely on the restriction of T and u^α within the past lightcone of x . Furthermore, by dissipation, it is mostly affected by past events that are close to x , as it loses memory of the far past.

Starting from these observations, we proposed a simple explanation of why the gradient series (1) often diverges: In a nutshell, the construction of the gradient series amounts to interchanging the Taylor series of the fields $\{T, u^\alpha\}$ with the spacetime integral defining $T^{\mu\nu}[T, u^\alpha]$. This delicate exchange of limits is often not allowed. In fact, the n th term of a Taylor expansion diverges like $\sim t^n$ at past infinity, and the usual theorems for exchanging limits and integrals (like the dominated convergence theorem [67]) do not apply. In some cases, Borel resummation “brings the series back inside $T^{\mu\nu}$,” and it reconstructs the correct functional $T^{\mu\nu}[T, u^\alpha]$.

So, when does hydrodynamics break down? From our analytical example, one can give two answers, depending on the definition of the word “hydrodynamics.”

Definition 1: Broad.—Hydrodynamics is the ability to express the conserved currents in terms of the fluid variables alone, e.g., $T^{\mu\nu} = T^{\mu\nu}[T, u^\alpha]$.

Using this definition, we find that hydrodynamics never breaks down, at any Kn , provided we know the near past of the fluid (“near past” \approx “few mean free paths ago”). Unfortunately, according to Definition 1, solving hydrodynamics amounts to solving $\partial_\mu T^{\mu\nu}[T, u^\alpha] = 0$, which is, in general, a convoluted integrodifferential equation.

Definition 2: Narrow.—Hydrodynamics is the ability to express the conserved currents as local functions of the

fluid fields and a finite number of derivatives thereof, e.g., $T^{\mu\nu} = T^{\mu\nu}(T, u^\alpha, \partial T, \partial u^\alpha, \dots, \partial^n T, \partial^n u^\alpha)$.

In our model, nature conspires to break the validity of the whole gradient expansion at the same breakdown scale of Navier-Stokes ($\text{Kn} \sim 0.1$). Thus, higher derivative corrections do not expand the applicability of the gradient expansion beyond first order.

This work was partially supported by a Vanderbilt Seeding Success Grant. We also thank M. Disconzi and J. Noronha for reading the manuscript and providing useful feedback.

-
- [1] J. Noronha-Hostler, J. Noronha, and M. Gyulassy, *Nucl. Phys. A* **956**, 890 (2016).
- [2] M. Strickland, *Acta Phys. Pol. B* **45**, 2355 (2014).
- [3] M. P. Heller and M. Spaliński, *Phys. Rev. Lett.* **115**, 072501 (2015).
- [4] M. Alqahtani, M. Nopoush, and M. Strickland, *Prog. Part. Nucl. Phys.* **101**, 204 (2018).
- [5] P. Romatschke, *J. High Energy Phys.* **12** (2017) 079.
- [6] P. Romatschke and U. Romatschke, *Relativistic Fluid Dynamics In and Out of Equilibrium* (Cambridge University Press, Cambridge, England, 2019).
- [7] P. Romatschke, *Phys. Rev. Lett.* **120**, 012301 (2018).
- [8] W. Florkowski, M. P. Heller, and M. Spaliński, *Rep. Prog. Phys.* **81**, 046001 (2018).
- [9] G. S. Denicol and J. Noronha, *Phys. Rev. Lett.* **124**, 152301 (2020).
- [10] M. P. Heller, A. Serantes, M. Spaliński, V. Svensson, and B. Withers, *Phys. Rev. D* **104**, 066002 (2021).
- [11] E. R. Most, J. Noronha, and A. A. Philippov, *Mon. Not. R. Astron. Soc.* **514**, 4989 (2022).
- [12] E. R. Most, S. P. Harris, C. Plumberg, M. G. Alford, J. Noronha, J. Noronha-Hostler, F. Pretorius, H. Witek, and N. Yunes, *Mon. Not. R. Astron. Soc.* **509**, 1096 (2021).
- [13] M. P. Heller, A. Serantes, M. Spaliński, V. Svensson, and B. Withers, *Phys. Rev. X* **12**, 041010 (2022).
- [14] C. Plumberg, D. Almaalol, T. Dore, J. Noronha, and J. Noronha-Hostler, *Phys. Rev. C* **105**, L061901 (2022).
- [15] N. Mullins, G. S. Denicol, and J. Noronha, *Phys. Rev. D* **106**, 056024 (2022).
- [16] D. Wagner and L. Gavassino, *Phys. Rev. D* **109**, 016019 (2024).
- [17] L. Gavassino and J. Noronha, *Phys. Rev. D* **109**, 096040 (2024).
- [18] L. Gavassino, *Entropy* **26**, 147 (2024).
- [19] I. Aniceto, J. Noronha, and M. Spaliński, [arXiv:2401.06750](https://arxiv.org/abs/2401.06750).
- [20] J. Noronha, B. Schenke, C. Shen, and W. Zhao, [arXiv:2401.09208](https://arxiv.org/abs/2401.09208).
- [21] N. G. Hadjiconstantinou, *Phys. Fluids* **18**, 111301 (2006).
- [22] G. S. Denicol, H. Niemi, E. Molnár, and D. H. Rischke, *Phys. Rev. D* **85**, 114047 (2012).
- [23] K. Huang, *Statistical Mechanics*, 2nd ed. (John Wiley & Sons, New York, 1987).
- [24] H. Struchtrup and P. Taheri, *IMA J. Appl. Math.* **76**, 672 (2011).
- [25] G. S. Rocha, G. S. Denicol, and J. Noronha, *Phys. Rev. D* **106**, 036010 (2022).
- [26] G. S. Rocha, D. Wagner, G. S. Denicol, J. Noronha, and D. H. Rischke, *Entropy* **26**, 189 (2024).
- [27] R. Baier, P. Romatschke, D. Thanh Son, A. O. Starinets, and M. A. Stephanov, *J. High Energy Phys.* **04** (2008) 100.
- [28] S. c. v. Grozdanov, P. K. Kovtun, A. O. Starinets, and P. Tadić, *Phys. Rev. Lett.* **122**, 251601 (2019).
- [29] P. Kovtun, *J. High Energy Phys.* **10** (2019) 034.
- [30] H. Struchtrup, *Continuum Mech. Thermodyn.* **17**, 43 (2005).
- [31] M. P. Heller, A. Serantes, M. Spaliński, and B. Withers, [arXiv:2305.07703](https://arxiv.org/abs/2305.07703).
- [32] M. P. Heller, A. Serantes, M. Spaliński, V. Svensson, and B. Withers, *Phys. Rev. Lett.* **128**, 122302 (2022).
- [33] P. Kovtun, G. D. Moore, and P. Romatschke, *Phys. Rev. D* **84**, 025006 (2011).
- [34] G. S. Denicol and J. Noronha, [arXiv:1608.07869](https://arxiv.org/abs/1608.07869).
- [35] G. C. Pomraning, *The Equations of Radiation Hydrodynamics* (Dover Publications, Mineola, 1973).
- [36] D. Mihalas and B. Weibel Mihalas, *Foundations of Radiation Hydrodynamics* (Oxford University Press, Oxford, 1984).
- [37] B. D. Farris, T. K. Li, Y. T. Liu, and S. L. Shapiro, *Phys. Rev. D* **78**, 024023 (2008).
- [38] M. Shibata, K. Kiuchi, Y.-i. Sekiguchi, and Y. Suwa, *Prog. Theor. Phys.* **125**, 1255 (2011).
- [39] A. Sadowski, R. Narayan, A. Tchekhovskoy, and Y. Zhu, *Mon. Not. R. Astron. Soc.* **429**, 3533 (2013).
- [40] L. Gavassino, M. Antonelli, and B. Haskell, *Symmetry* **12**, 1543 (2020).
- [41] D. Radice and S. Bernuzzi, *Astrophys. J.* **959**, 46 (2023).
- [42] L. H. Thomas, *Q. J. Math.* **1**, 239 (1930).
- [43] S. Weinberg, *Astrophys. J.* **168**, 175 (1971).
- [44] N. Udey and W. Israel, *Mon. Not. R. Astron. Soc.* **199**, 1137 (1982).
- [45] K. S. Thorne, *Mon. Not. R. Astron. Soc.* **194**, 439 (1981).
- [46] J. L. Anderson and H. R. Witting, *Physica (Amsterdam)* **74**, 466 (1974).
- [47] L. Rezzolla and O. Zanotti, *Relativistic Hydrodynamics*, edited by L. Rezzolla and O. Zanotti (Oxford University Press, New York, 2013), ISBN-10: 0198528906; ISBN-13: 978-0198528906.
- [48] G. S. Denicol and J. Noronha, *Phys. Lett. B* **850**, 138487 (2024).
- [49] K. Dusling, G. D. Moore, and D. Teaney, *Phys. Rev. C* **81**, 034907 (2010).
- [50] M. Srednicki, *Quantum Field Theory* (Cambridge University Press, Cambridge, England, 2007).
- [51] L. Gavassino, [arXiv:2404.12327](https://arxiv.org/abs/2404.12327).
- [52] G. S. Rocha, G. S. Denicol, and J. Noronha, *Phys. Rev. Lett.* **127**, 042301 (2021).
- [53] D. D. Clayton, *Principles of Stellar Evolution and Nucleosynthesis* (The University of Chicago Press, Chicago, 1983).
- [54] G. D. Moore, *J. High Energy Phys.* **05** (2018) 084.
- [55] G. S. Rocha, I. Danhoni, K. Ingles, G. S. Denicol, and J. Noronha, [arXiv:2404.04679](https://arxiv.org/abs/2404.04679).
- [56] W. Israel and J. Stewart, *Ann. Phys. (N.Y.)* **118**, 341 (1979).

- [57] F. Becattini, *Acta Phys. Pol. B* **47**, 1819 (2016).
- [58] L. Gavassino, *Classical Quantum Gravity* **38**, 21LT02 (2021).
- [59] L. Gavassino, *Classical Quantum Gravity* **39**, 185008 (2022).
- [60] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Vol. 2: Partial Differential Equations* (John Wiley and Sons, New York, 1989).
- [61] J. Rauch, *Partial Differential Equations*, Graduate Texts in Mathematics (Springer, New York, NY, 1991).
- [62] L. Gavassino, *Phys. Rev. D* **107**, 065013 (2023).
- [63] L. Gavassino, *Phys. Rev. X* **12**, 041001 (2022).
- [64] R. Geroch, [arXiv:gr-qc/0103112](https://arxiv.org/abs/gr-qc/0103112).
- [65] L. García-Colín, R. Velasco, and F. Uribe, *Phys. Rep.* **465**, 149 (2008).
- [66] R. M. Wald, *General Relativity* (Chicago University Press, Chicago, IL, 1984).
- [67] W. Rudin, *Principles of Mathematical Analysis, Third Edition* (McGraw-Hill, Inc., New York, 1973).
- [68] K. Dusling and T. Schäfer, *Phys. Rev. C* **85**, 044909 (2012).
- [69] A. Kurkela and U. A. Wiedemann, *Eur. Phys. J. C* **79**, 776 (2019).
- [70] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.133.032302> for some general considerations about the gradient expansion in linear-response theory. The Supplemental Material also includes Refs. [71–76].
- [71] J. Steffe, *Rheological Methods in Food Process Engineering*, 2nd ed. (Freeman Press, East Lansing, 1996).
- [72] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, NY, 1999).
- [73] M. P. Heller, A. Serantes, M. Spaliński, and B. Withers, *Phys. Rev. Lett.* **130**, 261601 (2023).
- [74] L. Gavassino, *Phys. Lett. B* **840**, 137854 (2023).
- [75] G. S. Denicol, J. Noronha, H. Niemi, and D. H. Rischke, *Phys. Rev. D* **83**, 074019 (2011).
- [76] L. Gavassino, *Classical Quantum Gravity* **40**, 165008 (2023).
- [77] J. D. Bjorken, *Phys. Rev. D* **27**, 140 (1983).
- [78] The principal value guarantees that the (unphysical) singularity at $t = L$ gives a finite contribution to the integral. When $L \gg \tau(E)$, we can neglect it, and the integral can be effectively stopped at $L - \tau(E)$.
- [79] H. Krants and S. Parks, *A Primer of Real Analytic Functions* (Springer Basel, Basel, 1992).
- [80] The nonlocal nature of equations involving an infinite series of derivatives is well known [81]. Take, e.g., the constitutive relation $\varphi(x) = e^{a\partial_x}\psi(x)$, which is equivalent to $\varphi(x) = \psi(x + a)$ [82].
- [81] L. Gavassino, M. Disconzi, and J. Noronha, *Phys. Rev. Lett.* **132**, 162301 (2024).
- [82] H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics*, 2nd ed. (Wiley, New York, NY, 1985).