Field Theory for Mechanical Criticality in Disordered Fiber Networks

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Strain-controlled criticality governs the elasticity of jamming and fiber networks. While the upper critical dimension of jamming is believed to be $d_u = 2$, non-mean-field exponents are observed in numerical studies of 2D and 3D fiber networks. The origins of this remains unclear. In this study we propose a minimal mean-field model for strain-controlled criticality of fiber networks. We then extend this to a phenomenological field theory, in which non-mean-field behavior emerges as a result of the disorder in the network structure. We predict that the upper critical dimension for such systems is $d_u = 4$ using a Gaussian approximation. Moreover, we identify an order parameter for the phase transition, which has been lacking for fiber networks to date.

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Strain-controlled rigidity transitions and criticality have been identified in systems ranging from the jamming of particle suspensions to fiber networks and extracellular matrices [1–6]. These systems demonstrate a transition from a floppy to a rigid phase with increasing applied strain. Similar physics also appears to govern shear thickening as a function of strain rate [7–10]. Both fiber networks and suspensions exhibit striking features of critical phenomena near the onset of rigidity, including power-law distributed forces [11], diverging nonaffinity [12–16], and critical slowing-down in dynamics [17,18]. Despite the similarities between jamming and fiber networks, however, a notable distinction lies in the nature of their critical exponents. While frictionless jamming is mean-field in dimension $d \geq 2$ [4,19], non-mean-field exponents have been reported for both 2D and 3D fiber networks [5,14,20–26]. These non-mean-field exponents in fiber networks hint at an underlying difference in the nature of the two transitions, and a theoretical understanding of this criticality is lacking.

Here, we present a phenomenological field theory for strain-controlled criticality of fiber networks. We first propose a minimal model that reproduces the mean-field behavior of the phase transition. We then extend this to a field theory that incorporates network disorder, with which we are able to calculate an anomalous critical exponent and identify the upper critical dimension $d_u = 4$. We show how reduced levels of disorder, and particularly hyperuniformity [27–29] with vanishing long-wavelength fluctuations leads to mean-field behavior. Based on this theory, we also propose and computationally verify an order parameter for this transition, which has been lacking to date.

Strain-controlled criticality.—We start by briefly summarizing previous observations on strain-controlled criticality of fiber networks. Fiber networks are modeled as athermal networks of interconnected segments [30–33]. The energy of the network can be written as sum of bending and stretching energy, governed by the stretch rigidity μ and the bending rigidity κ. Networks with coordination number or connectivity $Z < 2d$ exhibit a mechanical phase transition, which can be characterized by the network stiffness $K = \partial^2 E/\partial\gamma^2$, with E being the network elastic energy at a given strain γ. For central force networks ($\kappa = 0$), K is discontinuous at a critical strain $\gamma = \gamma_c$, with $K = 0$ for $\gamma < \gamma_c$ and $K \geq K_c > 0$ for $\gamma \geq \gamma_c^+$ [21,23,34,35], see Fig. 1(a). Above the critical point, the networks stiffen with $K - K_c \sim \mu\Delta\gamma^f$, where $\Delta\gamma = \gamma - \gamma_c$ is the relative strain with respect to γ_c. For finite κ, the elasticity becomes continuous and $K \sim \kappa|\Delta\gamma|^{-\lambda}$ is observed for $\Delta\gamma < 0$. The critical behavior can thus be described by two exponents f and ϕ, where $\phi = \lambda + f$ [5].

Minimal model.—We introduce a minimal model in 2D that exhibits characteristic features of strain-controlled phase transition. Consider a chain of two connected segments, see Fig. 1(b). Each segment is an elastic spring with

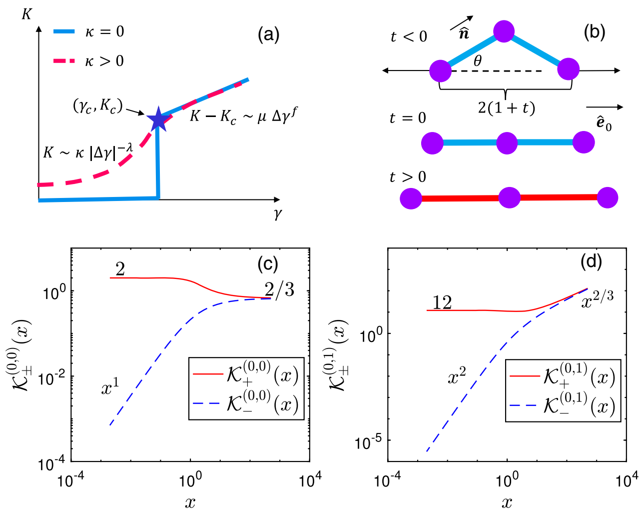

FIG. 1. (a) Schematic diagram of the scaling behavior of the stiffness K of fiber networks. (b) Minimal model for strain-controlled phase transition, formed by two connected segments under an extensile strain in the direction $\hat{\boldsymbol{e}}_0$. Red indicates tense segments for $t > 0$. (c) and (d) Scaling functions of the elasticity for the minimal model.

stretch rigidity $\mu = 1$ and rest length $\ell_0 = 1$. We deform the two ends of the chain in the horizontal direction $\hat{\boldsymbol{e}}_0$ with a reduced extensional strain $t = \epsilon - \epsilon_c$, which vanishes at the transition. By symmetry the middle node can only move in the vertical direction, leading to an angle θ between each segment and $\hat{\boldsymbol{e}}_0$. In two dimensions there is a degeneracy in the sign of θ. The Hamiltonian of the system is the total stretching energy, which is related to the extension of each segment, $\Delta\ell = (1+t)/\cos(\theta) - 1$,

$$H_m = \Delta\ell^2 \approx t^2 + t\theta^2 + \theta^4/4. \tag{1}$$

Here, we have kept the leading-order terms in t and θ. Equation (1) has a similar form as the Landau free energy [36] with t being a reduced-temperature-like parameter, and the minimum-energy solution is given by

$$\theta = \begin{cases} \pm\sqrt{-2t} & (t < 0) \\ 0 & (t \geq 0) \end{cases}. \tag{2}$$

The system energy reads $E^{(0)} = t^2\Theta(t)$ with Θ being the Heaviside function, suggesting that the system has a critical point $t = 0$: for $t < 0$ the chain buckles, indicative of a floppy phase, while for $t > 0$ the chain becomes straight and bears tension, corresponding to a rigid phase. For simple shear, we expect $t \propto \Delta\gamma$ to lowest order. As discussed below, however, nonlinear corrections to this can become important for comparison with experiments.

Next, we introduce the bending energy, which is modeled as harmonic energy that aligns segments to a particular angle θ_b. The Hamiltonian then reads

$$\begin{aligned} H_m &= t^2 + t\theta^2 + \theta^4/4 + \tilde{\kappa}(\theta - \theta_b)^2 \\ &\approx t^2 + t\theta^2 + \theta^4/4 - \kappa\theta, \end{aligned} \tag{3}$$

where we assume $\theta_b > 0$ without loss of generality and in view of the Z_2 symmetry. We assume $\kappa = 2\theta_b\tilde{\kappa}$ to be small. In Eq. (3) a constant term is neglected and only the leading term in θ is kept. An important assumption here is that θ in the relaxed state with respect to bending is in general different from θ in the critical state for $\kappa = 0$. This κ is similar to an external field in a Landau theory for ferromagnetism, but the alignment due to κ is not global, in contrast with an external aligning field.

Minimizing H_m we get [37]

$$E^{(0)} = |t|^2 \mathcal{E}_{\pm}^{(0)}(\kappa/|t|^{3/2}), \tag{4}$$

which reflects the mean-field criticality of fiber networks. To demonstrate this, we note that t is related to $\Delta\gamma$ by $t \propto \Delta\gamma + a(\Delta\gamma)^2$ to second order, where a is of order unity and depends on the deformation mode [39]. The differential modulus $K^{(0)} = \partial^2 E^{(0)}/\partial\gamma^2$ is then

$$K^{(0)} = |t|^0 \mathcal{K}_{\pm}^{(0,0)}(\kappa/|t|^{3/2}) + a|t|^1 \mathcal{K}_{\pm}^{(0,1)}(\kappa/|t|^{3/2}), \tag{5}$$

where the two scaling functions are found from the exact solution of the minimum energy state of Eq. (3)), see Figs. 1(c) and 1(d) [37]. Equation (5) reproduces the mean-field criticality of fiber networks sketched in Fig. 1(a): (i) In the rigid phase, $K^{(0)} - K_c \propto t^{f_0}$ for $\kappa = 0$ with $f_0 = 1$ and $K_c = 2$. (ii) In the floppy phase, $K^{(0)}$ vanishes for $\kappa = 0$ and is given by $K^{(0)} \sim \kappa|t|^{-\lambda_0}$ for $\kappa \ll |t|^{\lambda_0}$, where $\lambda_0 = 3/2$. These scaling exponents agree with previous mean-field derivations [34,35,40,41]. Moreover, we find that the stiffness at the critical point is different for $\kappa = 0$ and $\kappa \to 0$, with a ratio $(\lim_{\kappa\to 0} K)/K(\kappa = 0) = 1/3$. This difference and the scaling variable $\kappa/|t|^{3/2}$ are consistent with Ref. [35]. Because the second term in Eq. (5) contributes subdominantly to the elasticity, in what follows we assume $a = 0$ and set $t = \Delta\gamma$ below.

Field theory.—We now extend the minimal model to a field theory. Consider a fiber network in d dimensions with system size W and volume $V = W^d$, subject to an extensile strain with relative strain t in the direction $\hat{\boldsymbol{e}}_0$, representing an average direction of the developed force chain. Here we do not specify the exact deformation mode of the network and neglect the deformation in the transverse direction(s) of $\hat{\boldsymbol{e}}_0$, see Ref. [37] for detailed discussion. We divide the network into small blocks and approximate each block with a minimal model, see Fig. 2. The model parameter θ becomes a vector $\boldsymbol{m} = \hat{\boldsymbol{n}} - \hat{\boldsymbol{e}}_0$, with $\hat{\boldsymbol{n}}$ being the segment orientation of each minimal model.

We start with the rigid phase, whose Hamiltonian reads $H = \int \mathrm{d}\boldsymbol{x} h$ with

$$h = t^2 + t\boldsymbol{m}^2 + \boldsymbol{m}^4/4 - \boldsymbol{\kappa}\cdot\boldsymbol{m} + A(\nabla\boldsymbol{m})^2 + \boldsymbol{c}\cdot\boldsymbol{m}. \tag{6}$$

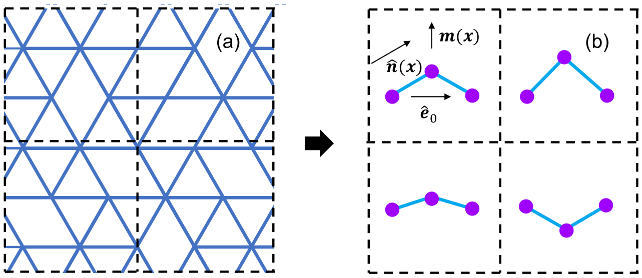

FIG. 2. Schematic diagram of the coarse-graining process of the field theory. (a) A fiber network is divided into small blocks. (b) The network region in each block is approximated by a minimal model.

Here, $\boldsymbol{m}(\boldsymbol{x})$ is a field that varies with the position $\boldsymbol{x}$ of the blocks. We use a phenomenological interaction of the form $A(\nabla \boldsymbol{m})^2 = A(\nabla_i m_j)^2$ that is lowest order in gradients and dominant at long wavelengths [42,43]. This represents the elastic cost of the relative deformation between blocks. In Eq. (6) we have also assumed that $\boldsymbol{m}$ is small near the critical point. Hence $\boldsymbol{m} = \hat{\boldsymbol{n}} - \hat{\boldsymbol{e}}_0$ is a $d-1$ dimensional transverse vector in the plane perpendicular to $\hat{\boldsymbol{e}}_0$, in which plane the effective bending rigidity $\boldsymbol{\kappa} = \kappa \hat{\boldsymbol{e}}_1$ can also be assumed to lie, with $\hat{\boldsymbol{e}}_1$ being the direction of the bending force. $\boldsymbol{c}(\boldsymbol{x})$ is a disordered field that is also transverse and emerges from the network quenched disorder, e.g., in the random cross-linking between fibers. $\boldsymbol{c} \cdot \boldsymbol{m}$ is the leading-order correction to the Hamiltonian due to such disorder, which is linear because disorder can locally break the rotational (Z_2 in 2D) symmetry of the minimal model. Importantly, both A and $\boldsymbol{c}$ are functions of κ and t, because they are phenomenological parameters that can change according to the network configuration.

The minimum-energy state is found by minimizing H with respect to $\boldsymbol{m}(\boldsymbol{x})$. For an ordered network with translational symmetry (e.g., a perfect lattice), $\boldsymbol{c} = 0$. Because thermal fluctuations are not present in athermal fiber networks (unlike the Ising model), the disorder-free model has a mean-field solution with an energy density $E/V = E^{(0)}$ (V is the volume) and an elastic modulus $K/V = K^{(0)}$ as given above. Real networks, however, are disordered with nonzero $\boldsymbol{c}$, which leads to spatial fluctuations of $\boldsymbol{m}$ in the minimum-energy state. We postulate that the disorder modifies the energy to $E/V \simeq E^{(0)} + E^{(1)}$. Here $E^{(0)}$ assumes the same scaling form as Eq. (4) (although the exact scaling function may be different), and

$$E^{(1)} = |t|^{2+f_1} \mathcal{E}_{\pm}^{(1)}(\kappa/|t|^{\phi_1}). \tag{7}$$

The exponent $2 + f_1$ ensures $K - K_c \sim t^{f_1}$ in the rigid phase. The complete elastic modulus reads $K/V \simeq K^{(0)} + K^{(1)}$, with

$$K^{(1)} = |t|^{f_1} \mathcal{K}_{\pm}^{(1)}(\kappa/|t|^{\phi_1}). \tag{8}$$

While Eq. (7) is a scaling ansatz, the validity of this scaling form has been verified in prior simulations [5,14,20,23]. $E^{(1)}$ results from the combination of two terms: the spatial correlating term and the disorder term [last two terms of Eq. (6), respectively]. It vanishes in the absence of disorder ($\boldsymbol{c} \to 0$) or when $A \to \infty$.

The rigid phase is stretch dominated, allowing us to take $\kappa = 0$ for convenience. For small t we expand A and $\boldsymbol{c}$ such that $A \simeq A_0$ and $\boldsymbol{c}(\boldsymbol{x}) \simeq \boldsymbol{c}_0(\boldsymbol{x}) + t\boldsymbol{c}_t(\boldsymbol{x})$. The finite stretching elasticity in the rigid phase suggests that there is a nonzero interaction strength A_0. On the other hand, $\boldsymbol{c}_0$ must be zero to ensure a critical point at $t = 0$, and the disordered field is characterized by $\boldsymbol{c}_t$ alone. For simplicity we take the disorder to be Gaussian and uncorrelated: $\langle c_{t,\alpha}(\boldsymbol{x}) c_{t,\beta}(\boldsymbol{x}') \rangle = C_t^2 \delta_{\alpha\beta} \delta(\boldsymbol{x} - \boldsymbol{x}')$.

To obtain the scaling exponent f_1, we expand Eq. (6) to quadratic order in the fluctuations (Gaussian approximation) [44]. We extract the scaling exponent using standard methods [44] and find $f_1 = d/2 - 1$ [37]. For $d \geq 4$, we have $f_1 \geq f_0$, such that $E^{(1)}$ cannot be distinguished from the mean-field behavior of $E^{(0)}$. Therefore, we predict the upper critical dimension $d_u = 4$, see Ref. [37]. For $d = 2$ and $d = 3$, we expect a non-mean-field exponent $f = f_1$. While the exact value of this exponent is likely to differ from the Gaussian approximation due to higher order contributions from the fluctuations, it is important that $f_1 < 1$ in $d < 4$, consistent with prior simulations [22–25]. The full Hamiltonian including the fourth order term is numerically minimized in 2D, which gives $f_1 \approx 0.43$ [37]. This is in reasonable agreement with previously reported exponents 0.34–0.55 for bulk and uniaxial strain [22,25]. Simulations find a dependence of the exponent on the mode of deformation and we do not expect our theory to accurately capture shear deformation since we only include deformation in a single direction $\hat{\boldsymbol{e}}_0$, see Ref. [37] for details. In addition, a hyperscaling relation of $E^{(1)}$ yields a correlation length $\xi \sim |t|^{-\nu}$, where $\nu = (2 + f_1)/d = 1/d + 1/2$ [14].

The floppy phase is more complicated. Because of the small or vanishing elasticity, the network can exhibit strong local strain fluctuations even in the absence of disorder. Therefore, we allow for the Hamiltonian density in Eq. (6) to be modified by a fluctuating mesoscopic strain field $\tilde{t}(\boldsymbol{x})$. To ensure a macroscopic relative strain t, we impose a set of constraints on $\tilde{t}$ for each $\mathcal{L}$, where $\mathcal{L}$ is any line that spans the network in the direction $\hat{\boldsymbol{e}}_0$: $(1/W) \int_{\mathcal{L}} ds \tilde{t}(\boldsymbol{x}(s)) = t$, i.e., the average relative strain for each $\mathcal{L}$ is t (see Fig. S3 for an illustration [37]). The network state is then found by energy minimization with respect to both $\tilde{t}$ and $\boldsymbol{m}$. We begin with central-force networks ($\kappa = 0$), which has zero elasticity in the floppy phase. This suggests that no interaction exists between blocks in the floppy phase, leading to $A = 0$. In this case, the Hamiltonian has infinite number of degenerate ground states for $t < 0$ if $\boldsymbol{c} = 0$: As long as $2\tilde{t}(\boldsymbol{x}) + \boldsymbol{m}^2(\boldsymbol{x}) = 0$ holds, the network has zero elastic energy. The deformation among these degenerate ground states corresponds to the floppy (zero) modes of real networks. Such a degeneracy vanishes for any nonzero $\boldsymbol{c}$, hence $\boldsymbol{c}$ must vanish for $\kappa = 0$. This degeneracy also precludes any perturbation-based method in solving the minimum-energy state, and an analytical identification of the exponents λ_1 and ϕ_1 is more challenging than in the rigid phase, see also Ref. [37]. For finite κ, an expansion in κ leads to $A = \kappa A_\kappa$ and $\boldsymbol{c}(\boldsymbol{x}) = \kappa \boldsymbol{c}_\kappa(\boldsymbol{x})$, where $\boldsymbol{c}_\kappa$ is assumed to be Gaussian and uncorrelated with $\langle c_{\kappa,\alpha}(\boldsymbol{x}) c_{\kappa,\beta}(\boldsymbol{x}')\rangle = C_\kappa^2 \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}')$. We numerically minimize the Hamiltonian in $d = 2$ and extract a non-mean-field $\lambda_1 = 1.88$ [37]. This is consistent with the values 1.85–1.98 obtained in previous simulations of 2D networks [22,23,25].

Order parameter.—An important quantity that has been missing in previous studies of fiber networks is the order parameter. Because of the local rotational symmetry of $\boldsymbol{m}$ (Z_2 symmetry in 2D) when $\kappa = 0$, we do not expect a magnetizationlike order parameter $\langle \boldsymbol{m} \rangle$. Instead, we consider $\langle \boldsymbol{m}^2 \rangle$ as the order parameter, which behaves as [37]

$$\langle \boldsymbol{m}^2 \rangle \sim \begin{cases} |t|^1 & (t < 0) \\ t^{1+f} & (t > 0). \end{cases} \tag{9}$$

Note that the scaling for $t < 0$ can also be found for $\langle \theta^2 \rangle$ of the minimal model. For real networks, we construct an order parameter with a similar form, $\langle \boldsymbol{m}^2 \rangle = \langle (\hat{\boldsymbol{n}} - \hat{\boldsymbol{n}}_c)^2 \rangle_{\tau_c}$. Here, $\hat{\boldsymbol{n}}$ is the orientation of each real segment at a given strain. $\hat{\boldsymbol{n}}_c$ is the segment orientation at the critical strain. The average is taken with respect to all segments in a network with weight τ_c, the magnitude of segment tension at the critical strain. This weight is introduced because not all segments are involved in the phase transition: The segments with $\tau_c = 0$ have no contribution to the elasticity. In Fig. 3(a) we show numerical results of $\langle \boldsymbol{m}^2 \rangle$ for 2D diluted triangular lattices, which are in excellent agreement with Eq. (9). While the nematic tensor $\underline{\underline{\boldsymbol{Q}}} = \langle \hat{\boldsymbol{n}}\hat{\boldsymbol{n}} - \underline{\underline{\boldsymbol{I}}}/d \rangle$ has been used to characterize nonlinear stiffening of fiber networks [45], we find that $\underline{\underline{\boldsymbol{Q}}}$ is featureless at the transition for fiber networks, as shown in Fig. S4 [37].

Discussion.—Although the mathematical form of the model we present is reminiscent of a Ginzburg-Landau theory, a crucial distinction lies in the role of temperature: In the Ginzburg-Landau theory thermal fluctuations play a crucial role, while for athermal fiber networks such thermal fluctuations are absent. The non-mean-field behavior in our model arises from quenched network disorder. While we analyze the effects of small uncorrelated disorder, disorder in real networks can be more complicated. Specific correlations in the disorder can alter critical exponents, as is the case in disordered Ising models [46]. Thus, it is not surprising that the exponents reported in previous numerical simulations depend on both network structure and deformation type, such that a universality class may not exist for fiber networks.

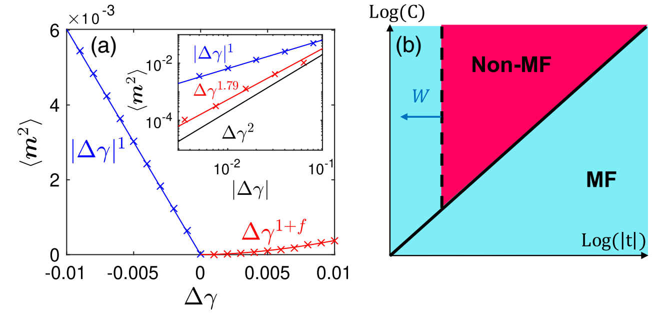

FIG. 3. (a) Simulation results (crosses) of the order parameter $\langle \boldsymbol{m}^2 \rangle$ for 2D central-force diluted triangular networks with connectivity $Z = 3.3$ and system size $W = 40$, subject to simple shear strain. Theoretical fitting according to Eq. (9) is shown in lines. The value of f is obtained from Ref. [23], $f = 0.79$. Inset: Scaling dependences of $\langle \boldsymbol{m}^2 \rangle$ for $\Delta\gamma < 0$ (blue) and $\Delta\gamma > 0$ (red). (b) Mean-field or non-mean-field diagram as function of the disorder C_t and the absolute relative strain $|t|$. The solid line indicates the Ginzburg criterion $|t| = t_G$ and the dashed line is the finite size criterion $|t| = t_W$, which decreases for increasing system size W, see Ref. [37] for further details.

Our work suggests that the upper critical dimension for fiber networks is $d_u = 4$, in contrast with numerical evidence of $d_u = 2$ for jamming, a superficially similar disordered rigidity transition. This difference may be due to differences in the nature of the disorder. Fiber networks have quenched permanent disorder that arises from the random cross-linking during network formation, while, the disorder in jamming is history-dependent and the local coordination is not fixed. It has been argued that the long wavelength disorder in jamming is suppressed close to the critical point, perhaps resulting in hyperuniformity [17]. To explore the effects of hyperuniformity in our system, we have calculated the exponent f in our model with hyperuniform disorder $\boldsymbol{c}(\boldsymbol{x})$, and find that it indeed reduces the upper critical dimension [37]. This may help to explain the difference between fiber networks and jamming.

Our results show that the energy and resulting mechanical quantities cannot strictly be described by a single scaling function [14,35]. However, the analysis above suggests that a recent scaling ansatz for the stiffness $K - K_c\Theta(t) \sim |t|^f \mathcal{K}_\pm(\kappa/|t|^\phi)$ involving a single scaling function and constant K_c in the rigid phase [23] should be accurate for most of the simulations to date [47]. Precisely at, or very close to the critical point, the mean-field $E^{(0)}$ and $K^{(0)}$ should be observed due to finite size effects, and these are consistent with recent scaling arguments for the critical point, as reported in Ref. [35]. Non-mean-field effects should only be observed within a certain range $t_W \lesssim |t| \lesssim t_G$, where $t_W \simeq W^{-1/\nu}$ represents the onset of finite-size effects where the correlation length becomes capped at the system size, see Fig. 3(b).

The upper bound is derived using the Ginzburg criterion, $t_G \approx C_t^{4/(4-d)} L^{-2d/(4-d)}$, where L is the average fiber length, see Ref. [37] for details.

The model here provides a theoretical framework for critical phenomena in biopolymer networks. While we focus on static criticality in athermal networks, extensions to thermal [48–50], dynamical [18] and possibly active [51,52] networks should be addressable with standard methods [53,54]. Moreover, due to the similarity between the criticality of fiber networks and jamming, it would be interesting to explore whether a similar field theory can be constructed for jamming.

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