Field Theory of the Fermi Function

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The Fermi function $F(Z,E)$ accounts for QED corrections to beta decays that are enhanced at either small electron velocity β or large nuclear charge Z. For precision applications, the Fermi function must be combined with other radiative corrections and with scale- and scheme-dependent hadronic matrix elements. We formulate the Fermi function as a field theory object and present a new factorization formula for QED radiative corrections to beta decays. We provide new results for the anomalous dimension of the corresponding effective operator complete through three loops, and resum perturbative logarithms and π enhancements with renormalization-group methods. Our results are important for tests of fundamental physics with precision beta decay and related processes.

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Introduction.—Many precision measurements and new physics searches involve charged leptons interacting with nucleons or nuclei. Examples include neutrino scattering to obtain fundamental neutrino parameters [1–6], muon-to-electron conversion to search for charged lepton flavor violation [7–10], and beta decay to measure fundamental constants [11–23] and search for new physics [24–34]. It is important to control radiative corrections to these processes [35–40]. The precision demands of superallowed nuclear beta decays are particularly stringent. As a consequence of conserved vector current relations, hadron and nuclear structure enter as small corrections. Experimental and nuclear uncertainties are being pushed to the level of 100 ppm [21,41], providing the best determination of the fundamental Cabibbo-Kobayashi-Maskawa (CKM) quark mixing parameter $|V_{ud}|$, and the most stringent low-energy constraint on scalar currents beyond the standard model [21]. In this Letter we present new results for long-distance QED corrections to beta decay [42–44] and discuss implications for the CKM unitarity discrepancy [21] and new physics constraints.

QED corrections are dramatically enhanced relative to naive power counting in the fine structure constant $\alpha \approx 1/137$ for large-Z nuclei and for small-β leptons (Z denotes the nuclear charge and β the lepton velocity). The Fermi function [45] in beta decay describes the enhancement (suppression) for negatively (positively) charged leptons propagating in a nuclear Coulomb field. For a nuclear charge Z and electron energy E it is traditionally defined by solving the Dirac equation in a pointlike Coulomb field. The result is then given as [45,46]

$$F(Z,E) = \frac{2(1+\eta)}{|\Gamma(2\eta+1)|^2}[\Gamma(\eta+\mathrm{i}\xi)]^2 \mathrm{e}^{\pi\xi}(2pr)^{2(\eta-1)}, \tag{1}$$

where $\eta \equiv \sqrt{1-(Z\alpha)^2}$, $\xi = Z\alpha/\beta$, $p = \sqrt{E^2 - m^2}$, and m is the electron mass. The quantity r denotes a short distance regulator identified approximately as the nuclear size [47]. Several questions arise in the application of $F(Z,E)$ to physical processes. (1) What is the scale r^{-1} and how does it relate to conventional renormalization in quantum field theory? (2) How can other radiative corrections be included systematically? (3) What is the relation between the Fermi function with $Z=1$ and the radiative correction to neutron beta decay? Answering these questions is important for the interpretation of precision beta decay experiments. For example, corrections at order $\alpha(Z\alpha)^2$ must be included at the current precision ($\sim 3\times 10^{-4}$) of $|V_{ud}|$ extractions [21]. These corrections require a theoretically self-consistent treatment of both the Fermi function and other radiative corrections, but have previously been treated only in a heuristic ansatz [38,48]. To answer these questions, we reformulate the Fermi function in effective field theory (EFT) and study its interplay with subleading radiative corrections.

Factorization and all-orders matching.—Factorization arises from the separation of different energy scales involved in a physical process [49–51]. Nuclear beta decays involve physics at the weak scale $\sim$100 GeV, the hadronic scale $\sim$1 GeV, the scale of nuclear structure $\Lambda_{\rm nuc} \sim 100$ MeV, and the kinematic scales relevant for beta decay $E \sim 1$ MeV. The methods of EFT allow for each scale to be treated separately and facilitate the calculation of higher-order radiative corrections. In a sequence of EFTs, the components of a factorization formula are identified with a corresponding sequence of matching coefficients and a final low-energy matrix element. In the context of nuclear beta decays, the long-distance (or outer) radiative corrections can be computed in the low-energy effective theory, while structure-dependent and short-distance (or inner) radiative corrections are absorbed into the Wilson coefficient. Real radiation is straightforwardly included [52].

Consider the corrections to a tree-level contact interaction with a relativistic electron in the final state. Ladder diagrams from a Coulomb potential with source charge $+Ze$ correct the tree-level amplitude $\mathcal{M}_{\rm tree}$ with explicit loop integrals given by (see Ref. [43] for more details)

$$\bar{u}(p)\mathcal{M} = \sum_{n=0}^{\infty}(Ze^2)^n \int \frac{\mathrm{d}^D L_1}{(2\pi)^D}\int \frac{\mathrm{d}^D L_2}{(2\pi)^D}\cdots\int \frac{\mathrm{d}^D L_n}{(2\pi)^D}\frac{1}{\mathbf{L}_1^2+\lambda^2}\frac{1}{(\mathbf{L}_1-\mathbf{p})^2-\mathbf{p}^2-\mathrm{i}0} \times \frac{1}{(\mathbf{L}_1-\mathbf{L}_2)^2+\lambda^2}\frac{1}{(\mathbf{L}_2-\mathbf{p})^2-\mathbf{p}^2-\mathrm{i}0}\cdots\frac{1}{(\mathbf{L}_{n-1}-\mathbf{L}_n)^2+\lambda^2}\frac{1}{(\mathbf{L}_n-\mathbf{p})^2-\mathbf{p}^2-\mathrm{i}0} \times \bar{u}(p)\gamma^0(\not{p}-\not{L}_1+m)\gamma^0(\not{p}-\not{L}_2+m)\cdots\gamma^0(\not{p}-\not{L}_n+m)\mathcal{M}_{\rm tree}. \quad (2)$$

Integrals are evaluated in dimensional regularization with $D = 3 - 2\epsilon$ dimensions, and we have included a photon mass λ to regulate infrared divergences [54].

In contrast to the nonrelativistic problem [55], the relativistic expression (2) is UV divergent beginning at two-loop order, indicating sensitivity to short-distance structure. The factorization theorem reads [43]

$$\mathcal{M} = \mathcal{M}_S(\lambda/\mu_S)\mathcal{M}_H(p/\mu_S, p/\mu_H)\mathcal{M}_{\rm UV}(\Lambda/\mu_H), \quad (3)$$

counting $p \sim m \sim E$ and where Λ denotes the scale of hadronic and nuclear structure. We retain separate factorization scales μ_S and μ_H for clarity; conventional single scale matrix elements are obtained by setting $\mu_S = \mu_H = \mu$. After $\overline{\rm MS}$ renormalization, to all orders in $Z\alpha$, the soft function is given by $\mathcal{M}_S = \exp\left[\mathrm{i}\xi\log(\mu_S/\lambda)\right]$ [56,57]. Our result for the hard function is new [43], and is given (again to all orders in $Z\alpha$) by [58]

$$\mathcal{M}_H = e^{(\pi/2)\xi+\mathrm{i}\phi_H}\frac{2\Gamma(\eta-\mathrm{i}\xi)}{\Gamma(2\eta+1)}\sqrt{\frac{\eta-\mathrm{i}\xi}{1-\mathrm{i}\xi\frac{m}{E}}}\sqrt{\frac{E+\eta m}{E+m}}\sqrt{\frac{2\eta}{1+\eta}}\left(\frac{2p}{e^{\gamma_{\rm E}}\mu_H}\right)^{\eta-1}\left[\frac{1+\gamma^0}{2}+\frac{E+m}{E+\eta m}\left(1-\mathrm{i}\xi\frac{m}{E}\right)\frac{1-\gamma^0}{2}\right], \quad (4)$$

where $\phi_H = \xi[\log(2p/\mu_S) - \gamma_{\rm E}] - (\eta-1)(\pi/2)$, γ^0 is a Dirac matrix, and $\gamma_{\rm E} \approx 0.577$ is the Euler constant.

The leading-in-Z radiative correction to unpolarized observables from the soft and hard functions is given by

$$\langle|\mathcal{M}_H|^2\rangle = F(Z,E)|_{r_H} \times \frac{4\eta}{(1+\eta)^2}, \quad (5)$$

where we define $r_H^{-1} = \mu_H e^{\gamma_{\rm E}}$. The angle brackets denote contraction with lepton spinors, $\mathcal{M}_H \to \bar{e}\mathcal{M}_H\gamma^0\nu_L$, sum over final state spins, and division by the same expression in the absence of Coulomb corrections. Note that there is a finite multiplicative correction relating the $\overline{\rm MS}$ hard function to $F(Z,E)$.

Effective operators and anomalous dimension.—The structure-dependent factor $\mathcal{M}_{\rm UV}$ appearing in Eq. (3) depends on the process of interest. Important examples are beta decay transitions $[A,Z] \to [A,Z+1]e^-\bar{\nu}_e$ or $[A,Z+1] \to [A,Z]e^+\nu_e$. Superallowed beta decays are governed by an EFT consisting of QED for electrons and heavy charged scalar fields [59–62],

$$\mathcal{L}_{\rm eff} = -\mathcal{C}(\phi_v^{[A,Z+1]})^*\phi_v^{[A,Z]}\bar{e}\not{v}(1-\gamma_5)\nu_e + \text{H.c.}, \quad (6)$$

where $\phi_v^{[A,Z]}$ denotes a heavy scalar with electric charge Z whose momentum fluctuations are expanded about $p^\mu = M_{[A,Z]}v^\mu$, with $v^\mu = (1,0,0,0)$ in the nuclear rest frame. For neutron decay, the EFT involves spin-1/2 heavy fields [59–62],

$$\mathcal{L}_{\rm eff} = -\bar{h}_v^{(p)}(\mathcal{C}_V\gamma^\mu + \mathcal{C}_A\gamma^\mu\gamma_5)h_v^{(n)}\bar{e}\gamma^\mu(1-\gamma_5)\nu_e + \text{H.c.}, \quad (7)$$

where $h_v^{(p)}$ and $h_v^{(n)}$ denote spin-1/2 heavy fields with electric charge 1 and 0, respectively. Matching to the EFT represented by Eq. (6) or Eq. (7), we identify the components of Eq. (3) in terms of operator coefficients and matrix elements: $\mathcal{M}_{\rm UV}$ is proportional to (a linear combination of) $\mathcal{C}_i$, while $\mathcal{M}_H$ and $\mathcal{M}_S$ give the hard and soft contributions to the EFT matrix element. In $\mathcal{M}_H$, at each order in α, the leading power of Z is given by the explicit expression (4), obtained from the amplitudes (2). In particular, the leading-in-Z anomalous dimension is obtained from the μ_H dependence of Eq. (4), cf. Eq. (9) below.

We may proceed to analyze the renormalization-group properties of weak-current operators in the EFT. Radiative corrections enhanced by large logarithms, $L \sim \log(\Lambda_{\rm nuc}/E)$, are determined by the anomalous dimensions of the operators in (6) and (7), which are spin-structure independent, i.e., $\gamma_A = \gamma_V = \gamma_{\mathcal{O}}$. Writing

$$\gamma_{\mathcal{O}} = \frac{d\log\mathcal{C}}{d\log\mu} = \sum_{n=0}^{\infty}\sum_{i=0}^{n+1}\left(\frac{\alpha}{4\pi}\right)^{n+1}\gamma_n^{(i)} Z^{n+1-i} \equiv \gamma^{(0)}(Z\alpha) + \alpha\gamma^{(1)}(Z\alpha) + \cdots, \tag{8}$$

we note several interesting all-orders properties: (i) Powers of Z greater than the power of α do not appear [63]. (ii) The leading series involving $(Z\alpha)^n$ sums to

$$\gamma^{(0)} = \sqrt{1-(Z\alpha)^2} - 1. \tag{9}$$

This result is obtained by differentiating Eq. (4) with respect to μ_H. (iii) At each order in perturbation theory, the leading and first subleading powers of Z are related [66]:

$$\gamma_{2n-1}^{(1)} = n\gamma_{2n-1}^{(0)}, \qquad \gamma_{2n}^{(2)} = n\gamma_{2n}^{(1)} \qquad (n \geq 1). \tag{10}$$

When $Z = 0$, the problem reduces to a heavy-light current operator. Using our new result for $\gamma_2^{(1)} = 16\pi^2(6 - \pi^2/3)$ [42] and property (10), the complete result through three-loop order at arbitrary Z is

$$\gamma_{\mathcal{O}} = \frac{\alpha}{4\pi}\gamma_0^{(1)} + \left(\frac{\alpha}{4\pi}\right)^2\left[-8\pi^2 Z(Z+1) + \gamma_1^{(2)}\right] + \left(\frac{\alpha}{4\pi}\right)^3\left[16\pi^2 Z(Z+1)\left(6-\frac{\pi^2}{3}\right) + \gamma_2^{(3)}\right], \tag{11}$$

where $\gamma_{n-1}^{(n)}$, $n = 1, 2, 3$, are known from the heavy quark literature [67]. Our result for $\gamma_2^{(1)}$ disagrees with Ref. [39], which did not include the full set of relevant diagrams at $O(Z^2\alpha^3)$ [42]. Note that properties (9) and (10) also determine the anomalous dimension at order $Z^4\alpha^4$ and $Z^3\alpha^4$.

Renormalization group analysis.—An important advantage of identifying the Fermi function as a field theory object is the ability to resum large logarithms, $\sim\log(\Lambda_{\rm nuc}/E)$, at high perturbative orders using renormalization-group methods. Consider the solution to the renormalization-group equation,

$$d\log\mathcal{C} = \frac{\gamma(\alpha)}{\beta(\alpha)}d\alpha, \tag{12}$$

where α is the $\overline{\rm MS}$ QED coupling (for one dynamical electron flavor) and $\beta = d\alpha/d\log\mu = -2\alpha[\beta_0\alpha/(4\pi) + \beta_1\alpha^2/(4\pi)^2 + \cdots]$ [70]. Expanding γ and β in powers of α and Z, then integrating, we obtain a systematic expansion for the ratio of the renormalized operator coefficient at different scales, $C(\mu_H)/C(\mu_L)$. Setting $\mu_H \sim \Lambda$ and $\mu_L \sim m$, we thus resum large logarithms $\log(\Lambda/m)$ [74]. Since the convergence of the series in α is influenced by Z, let us consider several regimes of Z.

(i) *Large Z asymptotics*. Consider a large Z nucleus, counting $\log^2(\Lambda/m) \sim \alpha^{-1}$ and $Z \sim \alpha^{-1}$. For example, we may consider beta decays of ^{210}Pb or ^{239}U. Through $O(\alpha^{1/2})$,

$$\log\left(\frac{C(\mu_L)}{C(\mu_H)}\right) = [-\gamma^{(0)}(Z\alpha_L)L] + \left[b_0\alpha_L L^2\frac{(Z\alpha_L)^2}{2\sqrt{1-(Z\alpha_L)^2}}\right] + \left[b_0^2\alpha_L^2 L^3\frac{(Z\alpha_L)^2(3-2(Z\alpha_L)^2)}{6(1-(Z\alpha_L)^2)^{3/2}} - \alpha_L L\gamma^{(1)}(Z\alpha_L)\right], \tag{13}$$

where $\alpha_{H,L} \equiv \alpha(\mu_{H,L})$, $L = \log(\mu_H/\mu_L)$, and $b_0 = -\beta_0/(2\pi)$. Consider separately the terms in $\gamma^{(1)}$ with odd and even powers of $(Z\alpha)$. Using Eq. (10),

$$\gamma_{\rm odd}^{(1)} = \frac{1}{2}\frac{\partial}{\partial(Z\alpha)}\gamma^{(0)} = \frac{-Z\alpha}{2\sqrt{1-(Z\alpha)^2}}. \tag{14}$$

The corresponding decay rate corrections involve (less the known $Z\alpha^2$ correction) [76]

$$\delta\frac{|C(\mu_L)|^2}{|C(\mu_H)|^2} - \alpha(Z\alpha)\log\frac{\Lambda}{E} = \alpha\log\frac{\Lambda}{E}\left[\frac{1}{2}(Z\alpha)^3 + \frac{3}{8}(Z\alpha)^5 + \cdots\right]. \tag{15}$$

The even series $\gamma_{\rm even}^{(1)}$ is determined through three-loop order by Eq. (11).

(ii) *Intermediate Z*. Consider a medium Z nucleus, counting $\log^2(\Lambda/m) \sim Z^2 \sim \alpha^{-1}$. This is relevant for superallowed beta decays contributing to $|V_{ud}|$ extraction, which range from $Z = 6$ (^{10}C) to $Z = 37$ (^{74}Rb). Through $O(\alpha^{3/2})$, the scale dependence is

$$\log\left(\frac{C(\mu_L)}{C(\mu_H)}\right) = \frac{\gamma_0^{(1)}}{2\beta_0}\left\{\left[\log\frac{a_H}{a_L} + \frac{Z^2\gamma_1^{(0)}}{\gamma_0^{(1)}}(a_H - a_L)\right] + \left[\frac{Z\gamma_1^{(1)}}{\gamma_0^{(1)}}(a_H - a_L)\right] + \left[\left(\frac{\gamma_1^{(2)}}{\gamma_0^{(1)}} - \frac{\beta_1}{\beta_0}\right)(a_H - a_L) + \left(\frac{Z^2\gamma_2^{(1)}}{\gamma_0^{(1)}} - \frac{\beta_1}{\beta_0}\frac{Z^2\gamma_1^{(0)}}{\gamma_0^{(1)}}\right)\frac{1}{2}(a_H^2 - a_L^2) + \frac{Z^4\gamma_3^{(0)}}{\gamma_0^{(1)}}\frac{1}{3}(a_H^3 - a_L^3)\right]\right\}, \quad (16)$$

where $a_{H,L} = \alpha(\mu_{H,L})/(4\pi)$ and the square brackets account for effects at order $\alpha^{1/2}$, α^1, $\alpha^{3/2}$, etc.

Achieving permille precision demands proper treatment of terms through resummed order $\alpha^{3/2}$. This result (16) replaces (and disagrees with) logarithmically enhanced contributions at order $Z^2\alpha^3$ in the "heuristic estimate" of Sirlin and Zucchini [78]. Using our new result for $\gamma_2^{(1)}$ [42] we compare to this heuristic estimate, and investigate the convergence of perturbation theory in Fig. 1. Here we fix μ_H and compute the product of $|C(\mu_L)/C(\mu_H)|^2$ and the squared operator matrix element at μ_L, varying μ_L as an estimate of perturbative uncertainty [79]. Normalizing to the leading Fermi function (known analytically to all orders) this quantity corresponds to the outer radiative correction appearing in the beta decay literature [cf. Eq. (11) of Supplemental Material [52]]. We note that Eq. (11) is in fact sufficient for a resummation of $C(\mu_H)/C(\mu_L)$ through $O(\alpha^2)$, although for practical applications one would also need currently unknown operator matrix elements at $O(Z\alpha^2)$ [80].

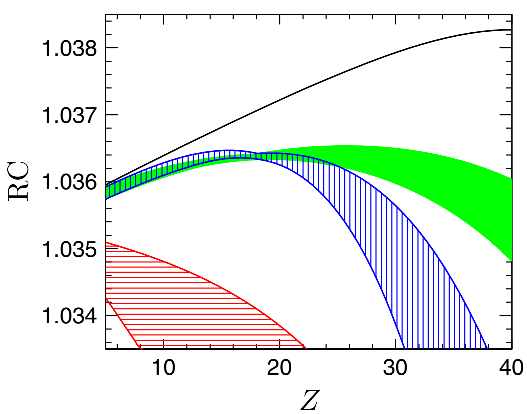

FIG. 1. Radiative correction to the beta decay rate as a function of nuclear charge, normalized to leading Fermi function. RC denotes the outer radiative correction, $1 + \delta'_R$, computed for fixed electron energy, cf. Eq. (11) of Supplemental Material [52]. Red, blue, and green curves show results correct through resummed order $\alpha^{1/2}$, α and $\alpha^{3/2}$, respectively. The black curve represents the central value for Sirlin's heuristic estimate as implemented in Ref. [21]. Illustrative values $E = 2$ MeV, $E_m = 5$ MeV, $\Lambda = 100$ MeV are used for the electron energy, maximum electron energy (which enters the one-loop matrix element [35]), and renormalization scale $\mu_H = \Lambda$, respectively. The width of the curves is given by varying $m_e/2 < \mu_L < 2E_m$. Analytic expressions can be obtained using Eq. (16) [52].

(iii) *Neutron beta decay.* Neutron beta decay corresponds to the case $Z = 0$ (in our convention); we therefore define $\gamma_{n-1} \equiv \gamma_{n-1}^{(n)}$. Again counting $\log^2(\Lambda/m) \sim \alpha^{-1}$, the resummation is [81]

$$\log\left(\frac{C(\mu_L)}{C(\mu_H)}\right) = \frac{\gamma_0}{2\beta_0}\left\{\log\frac{a_H}{a_L} + \left(\frac{\gamma_1}{\gamma_0} - \frac{\beta_1}{\beta_0}\right)(a_H - a_L)\right\}, \quad (17)$$

where the first term is of order $\alpha^{1/2}$, and the second term is of order $\alpha^{3/2}$. The complete result, correct through order $\alpha^{3/2}$, is obtained using (17) together with the one-loop low-energy matrix element.

Even after resumming logarithms in the ratio of hadronic and electron mass scales $\log(\Lambda/m)$, large coefficients remain in the perturbative expansion of the hard matrix element. While the class of amplitudes summed in the Fermi function are enhanced at small β and large Z, neither limit holds for neutron beta decay [83]. The large coefficients can instead be traced to an analytic continuation of the decay amplitude from spacelike to timelike values of momentum transfers. The enhancements are systematically resummed by renormalization of the hard factor $\mathcal{M}_H$ in the factorization formula (3) from negative to positive values of μ_S^2 (cf. Refs. [84,85]), with the result [44]

$$|\mathcal{M}_H(\mu_{S+}^2)|^2 = \exp\left[\frac{\pi\alpha}{\beta}\right]|\mathcal{M}_H(\mu_{S-}^2)|^2, \quad (18)$$

where $\mu_{S\pm}^2 = \pm 4p^2 - i0$ and $\mathcal{M}_H(\mu_{S-}^2)$ is free of π enhancements. This analysis systematically resums π-enhanced contributions, and does not rely on a nonrelativistic approximation.

Discussion.—At the outset of our discussion we posed three questions, which are now answered. (1) The scale r^{-1} appearing in the Fermi function (1) is unambiguously related to a conventional $\overline{\text{MS}}$ subtraction point, cf. Eq. (5). (2) The Fermi function is identified as the leading-in-Z contribution to the matrix element from the effective Lagrangian (6). Other radiative corrections are systematically computed using the same Lagrangian. (3) Numerically enhanced contributions in neutron beta decay arise from perturbative logarithms $|\log[(-\mathbf{p}^2 - i0)/\mathbf{p}^2]| = \pi$, and can be resummed to all orders. The result (18) differs from the nonrelativistic Fermi function ansatz [37,69] beginning at two-loop order.

Our EFT analysis allows us to systematically resum large perturbative logarithms and to incorporate corrections that are suppressed by $1/Z$ or E/Λ. New results include the following. (1) New coefficients in the expansion of the anomalous dimension for beta decay operators. We have computed the order $Z^2\alpha^3$ coefficient for the first time [86], and found a new symmetry linking leading-Z and subleading-Z terms in the perturbative expansion. Using our new result, and the existing heavy quark effective theory literature, we show that the first unknown coefficient occurs at four loops, at order $Z^2\alpha^4$ [42]. (2) New results for the large-Z asymptotics of QED radiative corrections to beta decay. We supply the infinite series of terms of order $\alpha(Z\alpha)^{2n+1}\log(\Lambda/E)$, replacing Wilkinson's ansatz [77], and present a new result for the term of order $\alpha(Z\alpha)^2\log(\Lambda/E)$, replacing Sirlin's heuristic estimate [38]. We provide the EFT matrix element to all orders in $Z\alpha$ and clarify its relation to the historically employed Fermi function [43]. (3) An all-orders resummation of "π-enhanced" terms in neutron beta decay, replacing the Fermi function ansatz. This substantially improves the convergence of perturbation theory and is important for modern applications to neutron beta decay [44].

Each of these results has important implications for ongoing and near-term precision beta decay programs [13,16,22,87–102]. Detailed computations are presented elsewhere [42–44]. Related work on new eikonal identities for charged current processes is presented in Ref. [64]. The same formalism applies to any situation involving charged leptons and nuclei, provided the lepton energy is small compared to the inverse nuclear radius.

An immediate conclusion of our study is that the existing estimate for $O(Z^2\alpha^3)$ corrections is incorrect. Focusing on the dominant logarithmically enhanced terms, the coefficient "a" in Sirlin's heuristic estimate [38,39] changes. For the 9 transitions with smallest $\mathcal{F}t$ uncertainty (at or below permille level), this leads to shifts ranging from 1.1×10^{-4} for ^{14}O to 1.4×10^{-3} for ^{54}Co [52], i.e., an order of magnitude larger than the estimated uncertainty on the outer radiative correction [21]. We observe that these shifts are comparable in magnitude to the CKM discrepancy, $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 - 1 = -0.0015(6)$ [21], and with a sign that goes in the direction of resolving the discrepancy. Accounting for these strongly Z-dependent corrections should also impact new physics constraints such as on scalar currents beyond the standard model [21]. A complete determination of the long-distance radiative corrections at the 10^{-4} level will require revisiting the $O(Z\alpha^2)$ matrix element in the pointlike EFT considered here; this work is ongoing. Future work will address factorization at subleading power and investigate the impact on phenomenology including hadronic [12,40] and nuclear [15] matching uncertainties.

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