Constructibility of AdS Supergluon Amplitudes

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We prove that all tree-level n-point supergluon (scalar) amplitudes in AdS_5 can be recursively constructed, using factorization and flat-space limit. Our method is greatly facilitated by a natural R-symmetry basis for planar color-ordered amplitudes, which reduces the latter to “partial amplitudes” with simpler pole structures and factorization properties. Given the n-point scalar amplitude, we first extract spinning amplitudes with $n-2$ scalars and one gluon by imposing “gauge invariance,” and then use a special “no-gluon kinematics” to determine the $(n+1)$-point scalar amplitude completely (which in turn contains the n-point single-gluon amplitude). Explicit results of up to 8-point scalar amplitudes and up to 6-point single-gluon amplitudes are included as Supplemental Material.

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Introduction.—Recent years have witnessed remarkable progress in computing and revealing new structures of holographic correlators, or “scattering amplitudes” in AdS space, at both tree [1–10] and loop [11–17] level. Although more focus has been on supergravity amplitudes in AdS, explicit results have also been obtained for “supergluon” tree amplitudes up to $n=6$ [18–22] in AdS super-Yang-Mills (sYM) theories (see Refs. [23–25] for loop-level results). In this Letter, we ask the interesting question about the “constructibility” of higher-point supergluon amplitudes purely from lower-point ones, and along the way we reveal nice structures for these amplitudes to all n.

The natural language for holographic correlators is the Mellin representation [26–28]. Mellin tree amplitudes are rational functions of Mellin variables. They can be determined by the residues at all physical poles (and pole at infinity encoded in the flat-space limit [22]), which for sYM are given by factorization with scalar and gluon exchanges [29]. These allowed the authors of [20,22] to bootstrap the supergluon amplitudes up to six point.

However, naively using factorization to bootstrap higher-point supergluon amplitudes is difficult, because we lack data of higher-point amplitudes involving spinning particles, which are needed to compute gluon-exchange contributions. We overcome this problem by getting “more” out of scalar-exchange contributions.

On one hand, we recognize a natural R-symmetry basis (Fig. 3) built from $SU(2)_R$ traces compatible with color ordering. Knowing lower-point scalar amplitudes, we are able to isolate the gluon-exchange contributions in factorization channels compatible with the trace structure. This enables us to extract the $(n-1)$-point single-gluon amplitude from the n-point scalar amplitude.

On the other hand, we identify certain “no-gluon kinematics” which is a consequence of the “gauge invariance” of single-gluon amplitudes. Regardless of the precise form of single-gluon amplitudes, at these special kinematic points, gluon exchanges are forbidden, imposing a powerful constraint on the amplitude.

Combining these two realizations, we devise a recursive algorithm (27) to obtain all-multiplicity supergluon tree amplitudes: start from the n-point scalar amplitude, extract from it the $(n-1)$-point single-gluon amplitude, and use these (sufficient) information to construct the $(n+1)$-point scalar amplitude. We include explicit results of up to 8-point scalar amplitudes and up to 6-point single-gluon amplitudes in the Supplemental Material [30].

Organization of Mellin amplitudes.—We are interested in the n-point supergluon amplitudes in AdS_5/CFT_4, which arise as the low energy description of many different theories [19,31–33]. For concreteness, consider the D3-D7-brane system in type IIB string theory in the probe limit (number

N_f of D7-branes much less than number N_c of D3-branes) [33]. On the world volume of D3-branes, we have an $\mathcal{N}=2$ SCFT, while on the world volume of D7-branes, gravity decouples at tree level and we have $\mathcal{N}=1$ sYM on $\mathrm{AdS}_5\times S^3$ [34]. The system has a symmetry $G_F=\mathrm{SU}(N_f)$ [35], which is global on the boundary and local in the bulk.

We study the connected correlator of half-BPS operators $\mathcal{O}^a(x,v)$ with dimension $\Delta=2$:

$$G_n^{(s)a_1\cdots a_n}=\langle\mathcal{O}^{a_1}(x_1,v_1)\cdots\mathcal{O}^{a_n}(x_n,v_n)\rangle, \tag{1}$$

$$\mathcal{O}^a(x,v)=\mathcal{O}^{a;\alpha_1\alpha_2}(x)v^{\beta_1}v^{\beta_2}\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}. \tag{2}$$

Here, $a_i=1,\ldots,\dim G_F$ are adjoint indices of G_F, and v^β $(\alpha_i,\beta_i=1,2)$ are auxiliary $\mathrm{SU}(2)_R$-spinors which extracts the R-spin-1 part of $\mathcal{O}^{a;\alpha_1\alpha_2}(x)$. The superscript $^{(s)}$ reminds us that $G_n^{(s)}$ is a correlator of scalar operators. For convenience, we also introduce the single-gluon correlators $G_n^{(v)}$ involving the Noether current $\mathcal{J}_\mu^a(x)$ of G_F, an $\mathrm{SU}(2)_R$-singlet with dimension $\Delta=3$:

$$G_{n;\mu}^{(v)a_1\cdots a_n}=\langle\mathcal{O}^{a_1}(x_1)\cdots\mathcal{O}^{a_{n-1}}(x_{n-1})\mathcal{J}_\mu^{a_n}(x_n)\rangle. \tag{3}$$

The bulk dual of $\mathcal{O}^a$ is ϕ_m^a for $m=1,2,3$ (supergluon), and the bulk dual of $\mathcal{J}_\mu^a$ is A_μ^a ("gluon"). Together, they compose the lowest Kaluza-Klein mode of the G_F gauge field on $\mathrm{AdS}_5\times S^3$. It can be shown that these are all the fields needed for $G_n^{(s)}$ at tree level [36].

The color decomposition for tree amplitudes in AdS space is identical to that for flat-space amplitudes [37]: we have color-ordered amplitudes as coefficients in front of traces of generators T^a in the adjoint representation:

$$G_n^{a_1\cdots a_n}=\sum_{\sigma\in S_{n-1}}\mathrm{tr}(T^{a_1}T^{a_2^\sigma}\cdots T^{a_{n-1}^\sigma}T^{a_n^\sigma})G_{1\sigma}, \tag{4}$$

where σ denotes a permutation of $\{2,\ldots,n\}$. Cyclic and reflection symmetry of the traces implies

$$G_{12\cdots n}=G_{2\cdots n1}=(-)^nG_{n\cdots 21}. \tag{5}$$

We will focus on $G_{12\cdots n}$ since any color-ordered amplitude can then be obtained by relabeling.

The natural language to describe such CFT correlators is the Mellin representation [26]. For scalar amplitudes,

$$G_{12\cdots n}^{(s)}=\int[\mathrm{d}\delta]\mathcal{M}_n^{(s)}(\{\delta_{ij}\},\{v_i\})\prod_{i<j}\frac{\Gamma(\delta_{ij})}{(-2P_i\cdot P_j)^{\delta_{ij}}}, \tag{6}$$

and for single-gluon amplitudes [29]:

$$G_{12\cdots n}^{(v)}=\int[\mathrm{d}\delta]\sum_{\ell=1}^{n-1}(Z_n\cdot P_\ell)\mathcal{M}_n^{(v)\ell}\prod_{i<j}\frac{\Gamma(\delta_{ij}+\delta_i^\ell\delta_j^n)}{(-2P_i\cdot P_j)^{\delta_{ij}+\delta_i^\ell\delta_j^n}}, \tag{7}$$

$$\text{where}\ \sum_{\ell=1}^{n-1}\delta_{\ell n}\mathcal{M}_n^{(v)\ell}=0. \tag{8}$$

Note that here δ_i^l is the Kronecker delta. We have used the embedding formalism following [29], where $P_i\cdot P_j=-\frac{1}{2}(x_i-x_j)^2$ and $Z_n\cdot P_\ell$ encodes the Lorentz tensor structure of $\mathcal{J}_\mu^a$. The Mellin variables are constrained as if $\delta_{ij}=p_i\cdot p_j$ for auxiliary momenta satisfying $\sum_i p_i=0$ and $p_i^2=-\tau_i=-2$, with conformal twist $\tau_i\coloneqq\Delta_i-J_i$ (J is the spin of an operator). Since $\mathcal{J}$ and $\mathcal{O}$ have the same twist, they are described by the same "kinematics."

Only the $\frac{1}{2}n(n-3)$ δ_{ij}'s are independent. Inspired by flat space [38], it proves convenient to introduce $\frac{1}{2}n(n-3)$ planar variables (with $\delta_{ii}\equiv-2$)

$$\mathcal{X}_{ij}\coloneqq 2+\sum_{i\le k,l<j}\delta_{kl}=2+\left(\sum_{i\le k<j}p_k\right)^2, \tag{9}$$

where we have $\mathcal{X}_{i,j}=\mathcal{X}_{j,i}$ with special cases $\mathcal{X}_{i,i+1}=0$ and $\mathcal{X}_{i,i}\equiv 2$. The inverse transform that motivated the associahedron in [38,39] reads

$$-2\delta_{ij}=\mathcal{X}_{i,j}+\mathcal{X}_{i+1,j+1}-\mathcal{X}_{i,j+1}-\mathcal{X}_{i+1,j}. \tag{10}$$

Planar variables correspond to n-gon chords (Fig. 1).

The planar variables are particularly suited for factorization [29] of color-ordered amplitudes. Since all relevant fields have $\tau=2$, schematically,

$$\mathcal{M}_{12\cdots n}\sim\frac{\mathcal{M}^{(m)}_{1\cdots(k-1)I}\mathcal{M}^{(m)}_{k\cdots nI}}{-(\mathcal{X}_{1k}+2m)},\qquad m=0,1,2,\ldots, \tag{11}$$

where a pole at $\mathcal{X}_{1k}=-2m$ corresponds to the exchange of a level-m descendant. By induction, all simultaneous poles of $\mathcal{M}_n$ consist of *compatible* planar variables (nonintersecting chords), which gives a (partial) triangulation of the n-gon dual to planar skeleton graphs (Fig. 1).

Another advantage of working with color-ordered amplitude is a natural basis for the R-charge structures. Let us define $\mathrm{SU}(2)_R$ trace as $V_{i_1i_2\cdots i_r}\coloneqq\langle i_1i_2\rangle\langle i_2i_3\rangle\cdots\langle i_ri_1\rangle$ where $\langle ij\rangle\coloneqq v_i^\alpha v_j^\beta\epsilon_{\alpha\beta}$. The Schouten identity $\langle ik\rangle\langle jl\rangle=\langle ij\rangle\langle kl\rangle+\langle il\rangle\langle jk\rangle$ enables us to expand any R structure to

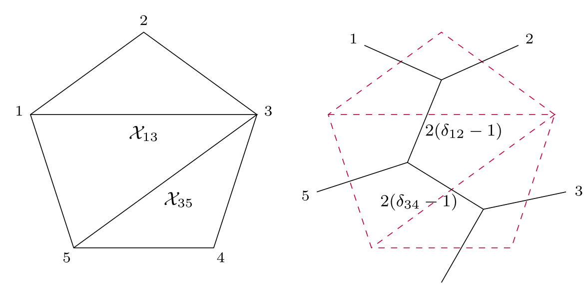

FIG. 1. Planar variables and dual skeleton graph for $n=5$.

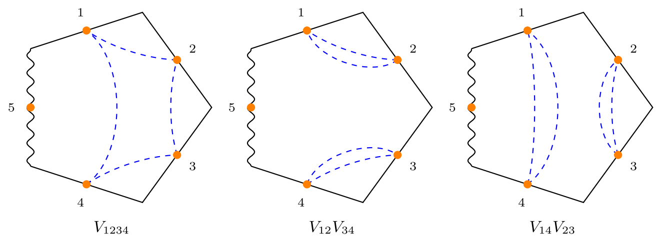

FIG. 2. $\mathcal{M}_5^{(v)}$ R structures.

products of noncrossing cycles or $\mathrm{SU}(2)_R$ traces:

$$\mathcal{M}_n^{(s)} = \sum_{\substack{\text{noncrossing}\\ \text{partition } \pi \\ \text{of } \{1,\ldots,n\}}} \left(\prod_{\text{cycle } \tau \in \pi} V_\tau \right) M_n^{(s)}(\pi), \tag{12}$$

$$\mathcal{M}_n^{(v)\ell} = \sum_{\substack{\text{noncrossing}\\ \text{partition } \pi \\ \text{of } \{1,\ldots,n-1\}}} \left(\prod_{\text{cycle } \tau \in \pi} V_\tau \right) M_n^{(v)\ell}(\pi). \tag{13}$$

For example, (Fig. 2)

$$\begin{aligned}
\mathcal{M}_4^{(s)} &= M_4^{(s)}(1234)V_{1234} \\
&\quad + M_4^{(s)}(12;34)V_{12}V_{34} + M_4^{(s)}(14;23)V_{14}V_{23}, \\
\mathcal{M}_4^{(v)\ell} &= M_4^{(v)\ell}(123)V_{123}, \\
\mathcal{M}_5^{(s)} &= M_5^{(s)}(12345)V_{12345} \\
&\quad + M_5^{(s)}(12;345)V_{12}V_{345} + \text{cyclic}, \\
\mathcal{M}_5^{(v)\ell} &= M_5^{(v)\ell}(1234)V_{1234} \\
&\quad + M_5^{(v)\ell}(12;34)V_{12}V_{34} + M_5^{(v)\ell}(14;23)V_{14}V_{23}.
\end{aligned}$$

Because a length-L trace picks up $(-)^L$ under reflection, for scalar amplitudes this cancels the sign in (5) while for single-gluon amplitudes the net result is a minus sign:

$$\begin{aligned}
&M_4^{(s)}(12;34) \overset{\text{ref}}{=} M_4^{(s)}(21;43) \overset{\text{cyc}}{=} M_4^{(s)}(14;23), \\
&M_5^{(v)}(12;34) \overset{\text{ref}}{=} -M_5^{(v)}(21;43), \\
&M_5^{(v)}(12;34) \text{ unrelated to } M_5^{(v)}(14;23).
\end{aligned}$$

For scalar amplitudes with $n = 6$, 7, we additionally have triple-trace R structures, and for $n \geq 8$ we need quadruple-trace R structures. The number of linearly independent R structures for $\mathcal{M}_n^{(s)}$ or $\mathcal{M}_{n+1}^{(v)}$ is $r_n = 1, 3, 6, 15, 36, 91, \ldots$ (Riordan numbers [40]).

Properties of Mellin amplitudes.—Factorization: Different exchanged fields contribute to different R structures. For a given channel, say $\mathcal{X}_{1k}$, we distinguish the compatible R structures π (none of the cycles τ intersect $\mathcal{X}_{1k}$) from the incompatible ones (Fig. 3). For scalar exchanges, (11) reads

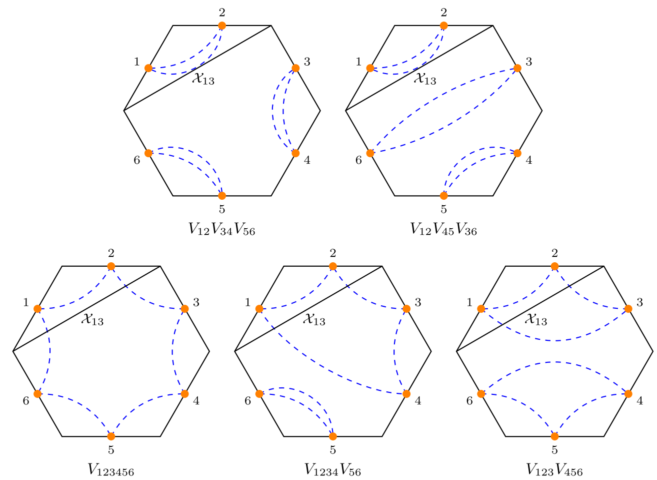

FIG. 3. $\mathcal{M}_6^{(s)}$ R structures compatible (above) and incompatible (below) with $\mathcal{X}_{13}$.

$$\underset{\mathcal{X}_{1k}=-2m}{\overset{(s)}{\mathrm{Res}}} \mathcal{M}_n^{(s)} = \mathcal{N}_s^{(m)} \texttt{glueR}\left(\mathcal{M}_{1\cdots(k-1)I}^{(s)(m)} \mathcal{M}_{k\cdots nI}^{(s)(m)} \right). \tag{14}$$

Here, $\mathcal{N}_s^{(m)} = 2$, and $\mathcal{M}_{1\cdots(k-1)I}^{(s)(m)}$ is a shifted version of the scalar amplitude $\mathcal{M}_{1\cdots(k-1)I}^{(s)}$:

$$\mathcal{M}_{1\cdots(k-1)I}^{(s)(m)} = \sum_{\substack{n_{ab}\geq 0 \\ \sum n_{ab}=m}} \mathcal{M}_{1\cdots(k-1)I}^{(s)}(\delta_{ab}+n_{ab}) \prod_{1\leq a<b<k} \frac{(\delta_{ab})_{n_{ab}}}{n_{ab}!}. \tag{15}$$

$\mathcal{M}_{k\cdots nI}^{(s)(m)}$ is defined similarly. The operation `glueR` glues together the traces. Note that there is the 1-1 correspondence of R structures in amplitudes and the operator product expansion (OPE):

$$\begin{gathered}
\langle \mathcal{O}(v_I)\mathcal{O}\cdots\mathcal{O}\rangle \supset \text{something} \times V_{ia\cdots bjI} \\
\Updownarrow \\
\mathcal{O}\cdots\mathcal{O} \supset \text{something} \times \langle ia\rangle\cdots\langle bj\rangle v_i^{(\alpha}v_j^{\beta)}\mathcal{O}_{\alpha\beta}
\end{gathered}$$

Since $\langle \mathcal{O}_{\alpha\beta}\mathcal{O}_{\gamma\delta}\rangle = \frac{1}{2}(\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma})$, we have

$$v_i^{(\alpha}v_j^{\beta)}v_k^{(\gamma}v_l^{\delta)}\langle \mathcal{O}_{\alpha\beta}\mathcal{O}_{\gamma\delta}\rangle = \langle il\rangle\langle jk\rangle - \frac{1}{2}\langle ij\rangle\langle lk\rangle, \tag{16}$$

which implies the following gluing rule:

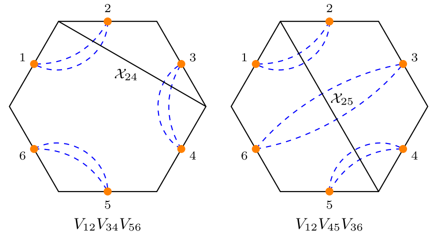

FIG. 4. Vanishing R structures.

$$\texttt{glueR}: V_{i\cdots jI} \otimes V_{Ik\cdots l} \mapsto V_{i\cdots jk\cdots l} - \frac{1}{2} V_{i\cdots j} V_{k\cdots l}. \quad (17)$$

We see that scalar exchanges contribute to both compatible and incompatible R structures. R structures with more than one cycle intersecting $\mathcal{X}_{1k}$ vanish (Fig. 4).

For gluon exchanges, (11) reads

$$\underset{\mathcal{X}_{1k}=-2m}{\overset{(v)}{\mathrm{Res}}} \mathcal{M}_n^{(s)} = \mathcal{N}_v^{(m)} \sum_{a=1}^{k-1} \sum_{i=k}^{n} \delta_{ai} \mathcal{M}^{(v)(m)a}_{1\cdots(k-1)I} \mathcal{M}^{(v)(m)i}_{k\cdots nI}. \quad (18)$$

Here, $\mathcal{N}_v^{(m)} = -[3/(1+m)]$, and $\mathcal{M}^{(v)(m)a}_{1\cdots(k-1)I}$ is $\mathcal{M}^{(v)a}_{1\cdots(k-1)I}$ shifted according to (15). We no longer need `glueR` because $\mathcal{J}$ is R neutral; gluon exchanges contribute to compatible R structures only.

An important consequence of gauge invariance (8) is that, at certain *no-gluon kinematics*, gluon exchanges are forbidden completely. To see this, let us denote $\mathcal{M}^{(v)(m)a}_{1\cdots(k-1)I} \equiv \mathcal{L}^{(m)a}$ and $\mathcal{M}^{(v)(m)i}_{k\cdots nI} \equiv \mathcal{R}^{(m)i}$, and solve $\mathcal{L}^{(m)1}, \mathcal{R}^{(m)k}$ using (8). The double sum in (18) becomes

$$\sum_{a=2}^{k-1} \sum_{i=k+1}^{n} \left(\delta_{ai} - \frac{\delta_{aI}}{\delta_{1I}} \delta_{1i} - \frac{\delta_{iI}}{\delta_{kI}} \delta_{ak} + \frac{\delta_{aI}\delta_{iI}}{\delta_{1I}\delta_{kI}} \delta_{1k} \right) \mathcal{L}^{(m)a} \mathcal{R}^{(m)i}.$$

If all $(k-2)(n-k)$ coefficients vanish on the support of $\mathcal{X}_{1k} = -2m$, gluon exchanges are forbidden, regardless of the detailed form of $\mathcal{L}^{(m)}$ and $\mathcal{R}^{(m)}$. The number of conditions equals the number of chords $\mathcal{X}_{ai}$ ($2 \le a \le k-1$ and $k+1 \le i \le n$) crossing $\mathcal{X}_{1k}$. Hence, the no-gluon conditions translate to $\mathcal{X}_{ai}$ taking special values $\mathcal{X}^*_{ai}$:

$$\mathcal{E}^{(m)}_{ai} \coloneqq \mathcal{X}_{ai} - \mathcal{X}^*_{ai} = 0, \quad (19)$$

$$X^*_{ai} = m - 1 + \frac{\mathcal{X}_{1a} + \mathcal{X}_{1i} + \mathcal{X}_{ak} + \mathcal{X}_{ik}}{2} + \frac{(\mathcal{X}_{1a} - \mathcal{X}_{ak})(\mathcal{X}_{1i} - \mathcal{X}_{ik})}{4(m+1)}. \quad (20)$$

Since gluon exchanges are forbidden at no-gluon kinematics, scalar exchanges alone fix the residue up to polynomials of $\mathcal{E}$'s:

$$\underset{\mathcal{X}_{1k}=-2m}{\mathrm{Res}} \mathcal{M}_n = \underset{\mathcal{X}_{1k}=-2m}{\overset{(s)}{\mathrm{Res}}} \mathcal{M}_n \Big|_{\mathcal{X}_{ai}=\mathcal{X}^*_{ai}} + \mathrm{poly}(\mathcal{E}^{(m)}_{ai}). \quad (21)$$

The special case of (18) where $k = n-1$ is particularly important. From the 3-point single-gluon amplitude [41]:

$$\mathcal{M}^{(v)(0)n-1}_{n-1,n,I} = \frac{i}{\sqrt{6}} V_{n-1,n}, \quad \mathcal{M}^{(v)(0)n}_{n-1,n,I} = -\frac{i}{\sqrt{6}} V_{n-1,n}, \quad (22)$$

we see that

$$\underset{\mathcal{X}_{1,n-1}=0}{\overset{(v)}{\mathrm{Res}}} \mathcal{M}_n^{(s)} = \frac{-3i}{\sqrt{6}} V_{n-1,n} \sum_{a=1}^{n-2} (\delta_{a,n-1} - \delta_{a,n}) \mathcal{M}^{(v)a}_{n-1}. \quad (23)$$

This is similar to the scaffolding relation in [42]. If we write the δ's in terms of $\mathcal{X}$'s, one can show that for each $2 \le a \le n-2$,

$$\mathcal{M}^{(v)a}_{n-1} - \mathcal{M}^{(v)a-1}_{n-1} = \frac{\partial}{\partial \mathcal{X}_{an}} \left(\frac{i\sqrt{2/3}}{V_{n-1,n}} \underset{\mathcal{X}_{1,n-1}=0}{\overset{(v)}{\mathrm{Res}}} \mathcal{M}_n^{(s)} \right). \quad (24)$$

Together with (8), these $(n-3)+1$ equations completely determine $\{\mathcal{M}^{(v)a}_{n-1}\}_{a=1}^{n-2}$. In other words, $(n-1)$-point single-gluon amplitudes can be extracted from the n-point scalar amplitude.

Flat space limit: It is shown in [19] that, with $\delta_{ij} = R^2 s_{ij}$, the leading terms of $\mathcal{M}_n^{(s)}$ in the limit $R \to \infty$ matches the flat space color-ordered n-gluon amplitude, with $\epsilon_i \cdot p_j = 0$ and $\epsilon_i \cdot \epsilon_j = \langle ij \rangle^2 = -V_{ij}$. Equivalently, this is the flat-space amplitude of $(n/2)$ pairs of scalars in Yang-Mills-scalar theory [43,44], which have been computed explicitly through $n = 12$. For even n, everything is clear, and $\mathcal{M}_n^{(s)} \sim \delta^{2-(n/2)}$. For example, with $\epsilon \cdot p = 0$,

$$\mathcal{A}_4^{\mathrm{flat}} = (\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4) \frac{s_{12} + s_{23}}{s_{12}} + (1 \leftrightarrow 3) - (\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4). \quad (25)$$

Using $V_{13}V_{24} = V_{12}V_{34} + V_{14}V_{23} - 2V_{1234}$ and writing $\mathcal{X}_{ij}$ in terms of δ_{ij}, we can check that this matches the leading terms of the correct $n = 4$ answer (up to overall normalization):

$$\mathcal{M}_4^{(s)} = 2\left(\frac{1}{\mathcal{X}_{13}} + \frac{1}{\mathcal{X}_{24}} - 1 \right) V_{1234} - \frac{2 + \mathcal{X}_{24}}{\mathcal{X}_{13}} V_{12}V_{34} - \frac{2 + \mathcal{X}_{13}}{\mathcal{X}_{24}} V_{14}V_{23}. \quad (26)$$

As an aside, it is a coincidence that the number $(n-1)!!$ of $(\epsilon \cdot \epsilon)^{(n/2)}$ terms equals r_n for $n = 4, 6$. For $n \ge 8$, these terms are not independent when translated to V. For odd n, the flat space amplitude vanishes due to the prescription $\epsilon_i \cdot p_j = 0$. The power counting $s^{2-(n/2)}$ means that the order $\delta^{2-\lfloor n/2 \rfloor}$ vanishes, and $\mathcal{M}_n^{(s)} \sim \delta^{2-\lceil n/2 \rceil}$. A more careful argument using the formula proposed in [27] leads to the same conclusion.

Constructing supergluon amplitudes.—It turns out that the properties and constraints satisfied by the Mellin amplitude discussed above are sufficient for a recursive construction of all tree-level supergluon amplitudes $\mathcal{M}_n^{(s)}$ for all n. Since $\mathcal{M}_{n-1}^{(v)}$ can be extracted from $\mathcal{M}_n^{(s)}$, we need only show that knowing $(\leq n-1)$-point scalar amplitudes and $(\leq n-2)$-point single-gluon amplitudes, we can construct the n-point scalar amplitude.

The proof starts by noticing that $(\leq n-2)$-point scalar and single-gluon amplitudes completely fix the residue of $\mathcal{M}_n^{(s)}$ on all poles $\mathcal{X}_{ij} = -2m$ with $\|i-j\| \geq 3$, where cyclic distance $\|i-j\| \coloneqq \min\{|i-j|, n-|i-j|\}$. Moreover, $(\leq n-1)$-point scalar amplitudes completely fix all incompatible channels. From these data, we can construct a rational function that can only differ from $\mathcal{M}_n^{(s)}$ by terms with only $\mathcal{X}_{i,i+2} = 0$ poles [45] and compatible traces. Then, we can write an ansatz for the possible difference, and completely fix it with constraints imposed by flat space limit and no-gluon kinematics.

Specifically, suppose $n = 2n' + 1$ is odd. Power counting $\mathcal{M}_n^{(s)} \sim \mathcal{X}^{1-n'}$, together with the fact that the ansatz only has $\mathcal{X}_{i,i+2} = 0$ poles and compatible channels, implies that the ansatz consists of terms of the form

$$\frac{\text{constant}}{\mathcal{X}^{n'-1}}.$$

The constants are fixed by scalar exchanges at no-gluon kinematics because the polynomial remainder in (21) is ruled out by power counting.

Suppose $n = 2n'$ is even. In the flat space limit, the leading terms are known, so the undetermined terms are subleading $\sim\mathcal{X}^{\leq 1-n'}$. Since there are at most n' simultaneous $\mathcal{X}_{i,i+2}$'s in the denominator, undetermined terms are of the form

$$\frac{\mathcal{X}^{\leq 1}}{\mathcal{X}^{n'}} \quad \text{or} \quad \frac{\text{constant}}{\mathcal{X}^{n'-1}}.$$

For $n \geq 6$, all such terms have no fewer than 2 simultaneous poles. To see that no-gluon kinematics is sufficient to fix the ansatz, simply note that we cannot construct a term $(\text{Numerator})/(\mathcal{X}_{ij}\mathcal{X}_{i'j'}\cdots)$ that vanishes at the no-gluon kinematics on every channel. For instance, for a term to vanish at no-gluon kinematics in both channels $\mathcal{X}_{13} = 0$ and $\mathcal{X}_{35} = 0$,

$$\begin{aligned} \text{Numerator} &= c_0\mathcal{X}_{13} + \sum_{i\neq 1,2,3} c_i\left(\mathcal{X}_{2i} + 1 - \frac{\mathcal{X}_{1i} + \mathcal{X}_{3i}}{2}\right) \\ &= d_0\mathcal{X}_{35} + \sum_{j\neq 3,4,5} d_j\left(\mathcal{X}_{4j} + 1 - \frac{\mathcal{X}_{3j} + \mathcal{X}_{5j}}{2}\right). \end{aligned}$$

Comparing both expressions, we see that these force Numerator $= 0$.

Therefore, from $\mathcal{M}_3^{(s)}$ and $\mathcal{M}_4^{(s)}$ (which contain contact terms and cannot be fixed by factorization), we can recursively construct $\mathcal{M}_n^{(s)}$ for all n as follows:

$$\cdots \rightsquigarrow \mathcal{M}_n^{(s)} \rightsquigarrow \mathcal{M}_{n-1}^{(v)} \rightsquigarrow \mathcal{M}_{n+1}^{(s)} \rightsquigarrow \cdots \qquad (27)$$

It is satisfying to see that 3- and 4-point interactions determine the amplitudes of all n, much like flat-space Yang-Mills-scalar theory. As a by-product, we also obtain $\mathcal{M}_n^{(v)}$. We emphasize that this is a constructive procedure, which is quite efficient (≤ 5 min to obtain $\mathcal{M}_8^{(s)}$).

Discussion and outlook.—Based on a better organization of R-symmetry structures which leads to a clear separation of scalar and gluon exchanges, we have shown that all-n supergluon tree amplitudes in AdS can be recursively constructed (27): we extract $(n-2)$-scalar-1-gluon amplitude from the n-scalar amplitude, which in turn determines the $(n+1)$-scalar amplitude. For instance, we could construct $\mathcal{M}_8^{(s)}$, knowing $\mathcal{M}_{\leq 7}^{(s)}$ and (hence) $\mathcal{M}_{\leq 6}^{(v)}$. In fact, we found in practice that even $\mathcal{M}_{\leq 5}^{(v)}$ suffices. Another observation is that $\mathcal{M}_{1\cdots(k-1)I}^{(s)(m)} = 0$ for $m \geq \lfloor k/2 \rfloor$, and $\mathcal{M}_{1\cdots(k-1)I}^{(v)(m)} = 0$ for $m \geq \lfloor (k-1)/2 \rfloor$, which explains the truncation of poles $\mathcal{X}_{ij} = -2m$ at $m \leq \lfloor \|j-i\|/2 \rfloor - 1$ in any $\mathcal{M}_n^{(s)}$. We will discuss these matters in detail in a forthcoming paper [46].

Our results provide more data for studying color-kinematics duality and double copy in AdS [9]. In addition, knowing the higher-point amplitudes, we can search for a set of Feynman rules. This will provide a better understanding of the bulk Lagrangian, as well as generalizing the Mellin space Feynman rules for scalars [47] and pure Yang-Mills [48,49].

Of course it would be highly desirable to apply similar methods to tree amplitudes with higher Kaluza-Klein modes ($\tau > 2$), and eventually at loop level. We are also very interested in adopting this method for bootstrapping supergravity amplitudes in AdS, as a generalization of the beautiful $n = 5$ results in [6,10]. Note that the R-symmetry basis and flat-space results [44] are available, and an immediate target would be the $n = 6$ supergravity amplitude.

We observe some universal behavior of our results, besides the "scaffolding" relation between a single gluon and a pair of scalars. For example, we find intriguing new structures such as "leading singularities," i.e., maximal residues, which take a form that resemble flat-space result in X variables. Our results and their generalizations strongly suggest that a possible combinatorial or geometric picture exists for AdS supergluon amplitudes, much like the scalar-scaffolding picture for gluons in flat space [42].

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[35] Other theories such as those arising in F theories [31,32] lead to different G_F, but otherwise the effective descriptions are the same.
[36] Within the supermultiplet containing $\mathcal{O}^a$ and $\mathcal{J}^a_\mu$, all other primaries are charged under $U(1)_r$, the Abelian part of the $\mathcal{N} = 2$ R-symmetry. Other half-BPS supermultiplets are dual to higher Kaluza-Klein modes, and they are charged under $SU(2)_L$ which is part of the isometry group $SO(4) = SU(2)_L \times SU(4)_R$ of S^3. The operators $\mathcal{O}^a$ and $\mathcal{J}^a_\mu$ are special in that they are neutral under $U(1)_r$ and $SU(2)_L$. Hence, all other fields can only appear in pairs and contribute at loop level.
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