Logarithmic Corrections to Kerr Thermodynamics

Daniel Kapec [1,2] Ahmed Sheta [1] Andrew Strominger [1] and Chiara Toldo [1,3]
[1] *Center for the Fundamental Laws of Nature, Harvard University, Cambridge, Massachusetts 02138, USA*
[2] *Harvard Center of Mathematical Sciences and Applications, Harvard University, Cambridge, Massachusetts 02138, USA*
[3] *Dipartimento di Fisica, Università di Milano, via Celoria 6, 20133 Milano MI, Italy*

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Recent work has shown that loop corrections from massless particles generate $\frac{3}{2}\log T_{\text{Hawking}}$ corrections to black hole entropy which dominate the thermodynamics of cold near-extreme charged black holes. Here we adapt this analysis to near-extreme Kerr black holes. Like $\text{AdS}_2 \times S^2$, the near-horizon extreme Kerr (NHEK) metric has a family of normalizable zero modes corresponding to reparametrizations of boundary time. The path integral over these zero modes leads to an infrared divergence in the one-loop approximation to the Euclidean NHEK partition function. We regulate this divergence by retaining the leading finite temperature correction in the NHEK scaling limit. This "not-NHEK" geometry lifts the eigenvalues of the zero modes, rendering the path integral infrared finite. The quantum-corrected near-extremal entropy exhibits $\frac{3}{2}\log T_{\text{Hawking}}$ behavior characteristic of the Schwarzian model and predicts a lifting of the ground state degeneracy for the extremal Kerr black hole.

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Introduction.—To an outside observer, a black hole appears to be an ordinary quantum mechanical system with finite entropy and highly chaotic internal dynamics. According to this picture, the exponential of the Bekenstein-Hawking entropy $e^{S_{\text{BH}}}$ represents the smooth (coarse-grained) leading approximation to the density of states of the black hole Hilbert space, whose average level spacing is expected to be $e^{-S_{\text{BH}}}$.

Although black holes are believed to be "ordinary" quantum mechanical systems, their thermodynamics is not generic. Black holes that spin or carry charge can be very large and very cold, and in the leading order semiclassical approximation to the black hole density of states there is an enormous ground state degeneracy e^{S_0} for these systems. In theories with unbroken supersymmetry, the existence of these ground states is sensible due to the huge degeneracy at zero coupling where there is enhanced symmetry, but in less symmetric models (like the black holes in our Universe) the degeneracy is surprising and one wants to know if it is merely the consequence of an approximation or not. This Letter addresses this question for Kerr black holes of spin J, which are extremal when $J = M^2$ with entropy $S_0 = 2\pi J$.

This question is related to another old puzzle about cold black holes [1]. For a black hole near extremality, semiclassical analysis predicts that the specific heat becomes order 1 at temperatures $T \sim J^{-3/2}$. Since the specific heat controls the size of thermodynamic fluctuations in nonequilibrium processes, below this temperature the emission of a thermal Hawking quantum causes large fluctuations in temperature and cannot be treated as a near-equilibrium process. The authors of [1] noted that the leading semiclassical approximation must receive large corrections at low temperatures and therefore cannot be trusted, but a derivation of the behavior in this regime was not given.

There are two proposed behaviors, both of which appear to have realizations in different models. The authors of [1] suggested that the black hole spectrum might have a gap $E_{\text{gap}} \sim J^{-3/2}$ above extremality, below which thermodynamics obviously no longer applies. For black holes with known microscopic descriptions (all of which are supersymmetric) this gap indeed exists [2–8].

The second possibility is that the large ground state degeneracy is an artifact of the leading order calculation, and that quantum corrections become more relevant at low temperatures and cause these states to spread out over a dense energy band above the vacuum. This is what one would naively expect for a nonsupersymmetric system like the Kerr black hole, the focus of this Letter. Although the exact spectrum of the black hole can only be computed nonperturbatively (the expected eigenvalue spacings in this part of the spectrum are $e^{-S_{\text{BH}}} \sim e^{-1/G_N}$), in this scenario one hopes to compute a perturbative correction to the density of states and determine whether or not $\rho_{\text{corr}}(E) \to 0$ as $E \to 0$.

Calculating this correction directly using the full Kerr geometry is a formidable task so far unachieved for any near-extremal black hole. However, there is another approach for studying the low temperature thermodynamics of spinning black holes that makes use of the emergent near-extremal throat (NHEK) and its approximate decoupling from the far region. At exact extremality the throat is infinitely long, and its asymptotic boundary serves as an effective stretched horizon for the black hole system. In this limit the far region decouples [9] and one expects that the relevant part of the black hole Hilbert space can be equivalently captured by gravitational dynamics in the throat according to the Kerr/CFT correspondence. The analogous formalism, when applied to spherically symmetric black holes, has led to precise matches of bulk gravitational calculations and microscopic counts [10,11].

However, using this approach to study the excited near-extremal microstates is subtle. Because of the strong backreaction effects present in low-dimensional systems with long-range forces, quantum gravity with exactly AdS_2 boundary conditions is believed to only describe ground states [16,17]. Calculations involving excited states are beset with infrared (IR) divergences, indicating a failure of the black hole to fully decouple from the far region.

In fact, as first noted by Sen [18], even the ground-state calculations can suffer from subtle divergences. The simplest IR divergence manifests in the one-loop correction to the Euclidean partition function in the extremal throat. In the process of calculating logarithmic corrections to extremal black hole entropy, Sen identified a set of normalizable zero modes in the NHEK throat corresponding to $\mathrm{Diff}(S^1)/SL(2,\mathbb{R})$ diffeomorphisms with noncompact support. Since these fluctuations are normalizable they must be integrated over, and since the domain of integration is infinite dimensional with no suppression the partition function diverges

$$Z_{\mathrm{NHEK}} \propto \int_{\mathrm{Diff}(S^1)/SL(2,\mathbb{R})} [Dh] = \infty. \tag{1}$$

The dependence of the measure on S_0 can be unambiguously determined, so these zero modes contribute a known logarithmic (in S_0) correction to the extremal entropy, assuming that it exists. However, the IR divergence of the partition function due to the unsuppressed fluctuations of the zero modes signals a subtlety in the calculation. As we will see, a proper treatment of these zero modes can remove the ground state degeneracy entirely. Instead of a system with tremendous entropy at zero temperature, one encounters a system with a dense energy band of e^{S_0} states spread out above the vacuum to which standard thermodynamics applies.

This resolution to the puzzle raised by (1) was first proposed for the analogous problem in the AdS_2 throat of extreme Reissner-Nordström in [7,19], and our analysis follows theirs closely. The strategy adopted in these papers amounts to turning on a small but finite temperature T, which necessitates the retention of subleading corrections to the metric in the near-extremal throat. These metric corrections lift the zero mode degeneracy and lead to $\log T$ corrections to the near-extremal entropy that agree with results derived using the Schwarzian model [20]. Laplace transforming this result to obtain the density of states, one finds that the ground state degeneracy vanishes. It is this prescription that we adapt for the near-extreme Kerr black hole, as described below.

The standard scaling limit into the throat of the extreme Kerr black hole takes the form

$$\hat{t} = \frac{t}{2\pi T}, \quad \hat{r} = r_+(T) + 4\pi r_0^2 T(r-1), \quad \hat{\phi} = \phi + \frac{t}{4\pi r_0 T} - t, \tag{2}$$

with $T \to 0$, leading to the decoupled NHEK metric

$$ds^2 = J(1+\cos^2\theta)\left(-(r^2-1)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2-1} + \mathrm{d}\theta^2\right) + J\frac{4\sin^2\theta}{1+\cos^2\theta}(\mathrm{d}\phi + (r-1)\mathrm{d}t)^2. \tag{3}$$

Here J is the spin, $r_+(T)$ is the radius of the outer horizon, r_0 is the radius of the extremal horizon and we take the limit in Boyer-Lindquist coordinates $(\hat{t}, \hat{r}, \theta, \hat{\phi})$. The Euclidean continuation of this metric has zero modes which lead to the infrared divergence (1). If one retains the leading $O(T)$ correction to this metric in the scaling limit (2), one obtains a distinct geometry which we will term the "not-NHEK" metric [21]

$$g_{\text{not-NHEK}} = g_{\mathrm{NHEK}} + \delta g, \tag{4}$$

with $\delta g \sim T$. Unlike (3), this metric is not an exact solution to the four-dimensional Einstein equation, although one can view it as a perturbative (in T) approximation to a solution whose nonlinear completion is the asymptotically flat finite temperature black hole. It is easy to see that the zero modes of (3) that lead to the divergence of (1) are lifted by the perturbation (4). The normalizable zero modes identified by Sen are metric deformations generated by non-normalizable diffeomorphisms with noncompact support, meaning that they can be written

$$h^{(n)} = \mathcal{L}_{\xi^{(n)}} g_{\mathrm{NHEK}} \tag{5}$$

for non-normalizable vector fields $\xi^{(n)}$. However, they are *not* diffeomorphisms of the not-NHEK metric

$$h^{(n)} \neq \mathcal{L}_{\zeta} g_{\text{not-NHEK}}. \tag{6}$$

They therefore acquire temperature-dependent eigenvalues at first order in perturbation theory. These perturbed eigenvalues can be used to obtain an approximation for the small-T (zeta-regularized) Euclidean partition function in the not-NHEK geometry. Interpreted as a correction to the black hole partition function, these new terms predict a lifting of the extremal ground state degeneracy for the Kerr black hole and a resolution of the puzzle described in [1].

There are many subtleties both in the calculation of logarithmic corrections to (near)-extreme black hole thermodynamics and in the physical interpretation of the results. There are both gauge and geometric ambiguities in the "gluing" of the decoupled near horizon geometry to the asymptotically flat region. The superradiant instability of the NHEK throat leads to travelling waves with imaginary conformal weights and a complex partition function, and calls into question the exact decoupling of the two regions, as does the nonexistence of a global vacuum for Kerr. Black holes in asymptotically flat spacetimes have finite lifetimes and are therefore metastable resonances rather than eigenstates. Since the "eigenvalue" spacing for these black holes is roughly e^{-S_0} while the lifetime is polynomial in S_0, the widths are naively much larger than the spacings and it is not clear whether it is sensible to discuss a discrete density of states.

While not all of these issues have been definitively settled, it is nevertheless clear that significant recent progress has been made in the understanding of logarithmic corrections to near-extreme charged black hole thermodynamics and extreme Kerr thermodynamics. The purpose of this Letter is to fill in the missing analysis of *near-extreme* Kerr thermodynamics by simply adopting both the assumptions and methodology used for the Reissner-Nordström case in the seminal papers [7,19]. Although the details differ, at the end we interestingly find a numerically identical entropy shift of $\frac{3}{2}\log T$, compatible with the Schwarzian dynamics [22–24]. Our main mathematical results are formulas for the finite temperature eigenvalues of the NHEK zero modes in the not-NHEK geometry (33), whose detailed form leads to the factor $\frac{3}{2}$ in (37).

Near-extreme Kerr.—The Kerr metric in Boyer-Lindquist coordinates is

$$ds^2 = -\frac{\Delta}{\Sigma}\left(\mathrm{d}\hat{t} - a\sin^2\theta\mathrm{d}\hat{\phi}\right)^2 + \frac{\Sigma}{\Delta}\mathrm{d}\hat{r}^2 + \Sigma\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[(\hat{r}^2+a^2)\mathrm{d}\hat{\phi} - a\mathrm{d}\hat{t}\right]^2, \tag{7}$$

with

$$\Delta(\hat{r}) = \hat{r}^2 - 2M\hat{r} + a^2, \qquad \Sigma(\hat{r},\theta) = \hat{r}^2 + a^2\cos^2\theta. \tag{8}$$

The spin of the black hole is given by $J = aM$, the inner and outer horizons occur at $r_\pm = M \pm \sqrt{M^2 - a^2}$, and the area of the outer event horizon is

$$A = 4\pi(r_+^2 + a^2) = 8\pi M\left(M + \sqrt{M^2 - (J/M)^2}\right). \tag{9}$$

The Hawking temperature and angular velocity of the horizon are

$$T = \frac{1}{4\pi M}\frac{\sqrt{M^2 - (J/M)^2}}{M + \sqrt{M^2 - (J/M)^2}}, \qquad \Omega_H = \frac{a}{2Mr_+}. \tag{10}$$

In the extremal limit $M^2 \to M_0^2 = J$, the horizons coalesce at $r_0 = M_0$, T vanishes, and $\Omega_H \to (1/2r_0)$.

At fixed angular momentum $J = r_0^2$, we parameterize small deviations from extremality by their temperature T. The relation (10) defines the thermodynamic energy $M(J,T)$, which has the small temperature expansion

$$M(T,J) = J^{1/2} + 4\pi^2 J^{3/2} T^2 + 32\pi^3 J^2 T^3 + 264\pi^4 J^{5/2} T^4 + \dots \tag{11}$$

Similarly, the horizons $r_\pm(T)$ at fixed T have the small temperature expansion

$$r_+(T) = J^{1/2} + 4\pi J T + 20\pi^2 J^{3/2} T^2 + 128\pi^3 J^2 T^3 + \dots,$$
$$r_-(T) = J^{1/2} - 4\pi J T - 12\pi^2 J^{3/2} T^2 - 64\pi^3 J^2 T^3 + \dots \tag{12}$$

The near-extremal entropy is then linear in T

$$S(T,J) = S_0 + 8\pi^2 J^{3/2} T + O(T^2), \tag{13}$$

and the average thermodynamic energy above extremality scales quadratically with temperature as

$$E(T,J) = M(T,J) - M_0 = 4\pi^2 J^{3/2} T^2 + O(T^3). \tag{14}$$

The NHEK throat: For a generic Kerr black hole, there is no meaningful geometric separation between the region of spacetime associated to the hole and the spacetime belonging to the far region: the two systems are coupled and the interactions between them cannot be ignored. The exception occurs when the black hole is near extremal, in which case a long throat of length $|\log T|$ develops just outside of the horizon. In the limit of infinite proper depth, this region is believed to approximately decouple from the far region, although this itself is a subtle statement. This region of spacetime is generally associated to the black hole.

In practice, it is possible to isolate the extremal throat by taking a scaling limit that zooms into the near horizon region of a family of cold Kerr geometries. The change of coordinates

$$\hat{t} = \frac{2r_0}{\varepsilon(T)}t, \qquad \hat{r} = r_+(T) + r_0\varepsilon(T)(\cosh\eta - 1),$$
$$\hat{\phi} = \phi + \frac{t}{\varepsilon(T)} - t, \qquad \varepsilon(T) = 4\pi r_0 T, \tag{15}$$

followed by the limit $T \to 0$, results in a spacetime that solves the Einstein equation in its own right:

$$ds^2 = J(1+\cos^2\theta)(-\sinh^2\eta dt^2 + d\eta^2 + d\theta^2) + J\frac{4\sin^2\theta}{1+\cos^2\theta}[d\phi + (\cosh\eta - 1)dt]^2. \tag{16}$$

This geometry, found by Bardeen and Horowitz in [26], is known as near-horizon extreme Kerr (NHEK) [27]. It is the analog of the Robinson-Bertotti universe obtained from the scaling limit of the near-extremal Reissner-Nordström black hole. The metric has $SL(2,\mathbb{R})\times U(1)$ symmetry with generators

$$L_{\pm 1} = \frac{e^{\mp t}}{\sinh\eta}[\cosh\eta\partial_t \pm \sinh\eta\partial_\eta + (\cosh\eta - 1)\partial_\phi],$$
$$L_0 = \partial_t + \partial_\phi, \qquad W = \partial_\phi. \tag{17}$$

It is commonly believed that at least part of the quantum mechanics of the Kerr black hole is captured by gravitational dynamics in this throat in analogy with the better-understood black holes with near-horizon AdS regions.

Quantum corrections to the throat thermodynamics.— There is to date no top-down microscopic construction of the four-dimensional Kerr black hole [28]. However, in accord with the usual assumptions we will identify the analytically continued gravitational path integral in the NHEK throat with the statistical partition function of the dual quantum mechanics. Following Sen [18] we analytically continue $t = -i\tau$ in (16) which gives

$$ds^2 = J(1+\cos^2\theta)(d\eta^2 + \sinh^2\eta d\tau^2 + d\theta^2) + \frac{4J\sin^2\theta}{1+\cos^2\theta}[d\phi - i(\cosh\eta - 1)d\tau]^2. \tag{18}$$

Regularity of the geometry at $\eta = 0$ requires the periodicity $\tau \sim \tau + 2\pi$. The partition function in the near-horizon region of Kerr is given formally by an integral over metrics subject to a certain set of boundary conditions [18]

$$Z = \int [Dg] e^{-I[g]}, \qquad I[g] = -\frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{g}R + I_{\text{boundary}}. \tag{19}$$

The boundary conditions and corresponding boundary terms in the action determine the statistical ensemble computed by the path integral.

The geometry (18) is a classical saddle-point for the integral (19) satisfying the appropriate boundary conditions, and therefore provides the leading approximation $Z \approx \exp(-I[g_{\text{NHEK}}])$ to the black hole partition function. In [18] it was shown that this saddle-point approximation, including the correct boundary contributions, reproduces the semiclassical entropy S_0 of the extremal Kerr black hole. However, the path integral (19) is not well-defined beyond the leading saddle point approximation: it is beset with UV divergences, and the instability of the NHEK throat due to superradiance means that any sensible definition of the integral will necessarily make Z complex. Nevertheless, in [18] Sen managed to extract some universal information about the dependence of (19) on S_0 through a careful analysis of the 1-loop determinant of massless fields on the background (18).

Quantum corrections to NHEK entropy and zero modes: The determination of the logarithmic corrections to the extremal Kerr entropy requires path-integration over the massless fields propagating on the NHEK throat. Expanding about the saddle-point $g = \bar{g} + h$, with $\bar{g}$ given by (18) and h a normalizable perturbation, the 1-loop approximation is controlled by the linearized kinetic operator for h

$$Z \approx \exp(-I[\bar{g}]) \int [Dh] \exp\left[-\int d^4x\sqrt{\bar{g}} h D[\bar{g}] h\right]. \tag{20}$$

Calculations are performed with the gauge fixing term

$$\mathcal{L}_{GF} = \frac{1}{32\pi}\bar{g}_{\mu\nu}\left(\bar{\nabla}_\alpha h^{\alpha\mu} - \frac{1}{2}\bar{\nabla}^\mu h^\alpha_\alpha\right)\left(\bar{\nabla}_\beta h^{\beta\nu} - \frac{1}{2}\bar{\nabla}^\nu h^\beta_\beta\right), \tag{21}$$

which, when combined with the Einstein-Hilbert action, yields the linearized kinetic term [31] for NHEK:

$$h_{\alpha\beta} D^{\alpha\beta,\mu\nu}_{\text{NHEK}} h_{\mu\nu} = -\frac{1}{16\pi} h_{\alpha\beta}\left(\frac{1}{4}\bar{g}^{\alpha\mu}\bar{g}^{\beta\nu}\bar{\Box} - \frac{1}{8}\bar{g}^{\alpha\beta}\bar{g}^{\mu\nu}\bar{\Box} + \frac{1}{2}\bar{R}^{\alpha\mu\beta\nu}\right) h_{\mu\nu}. \tag{22}$$

The determinant of this operator cannot be calculated exactly due to the reduced symmetry of the problem, but the terms contributing logarithmic corrections in S_0 can be extracted indirectly through the heat kernel expansion. Importantly, the operator appearing in (22) supports a family of normalizable zero modes

$$h^{(n)}_{\mu\nu} dx^\mu dx^\nu = \frac{1}{4\pi}\sqrt{\frac{3}{2}}\sqrt{|n|(n^2-1)}\frac{(1+\cos^2\theta)e^{in\tau}(\sinh\eta)^{|n|-2}}{(1+\cosh\eta)^{|n|}}\left(d\eta^2 + 2i\frac{n}{|n|}\sinh\eta d\eta d\tau - \sinh^2\eta d\tau^2\right), \quad |n| > 1, \tag{23}$$

which are not correctly accounted for by the heat kernel and which must be treated separately [32]. These metric perturbations are zero modes precisely because they are generated by large diffeomorphisms left unfixed by harmonic gauge (21). In other words, they obey $h^{(n)} \propto \mathcal{L}_{\xi^{(n)}} g_{\rm NHEK}$, with the vector field given by

$$\xi^{(n)} = e^{in\tau}\tanh^{|n|}(\eta/2)\left(-\frac{|n|(|n|+\cosh\eta)+\sinh^2\eta}{\sinh^2\eta}\partial_\tau + \frac{in(|n|+\cosh\eta)}{\sinh\eta}\partial_\eta + \frac{i(\cosh\eta+1+|n|-n^2)}{\cosh\eta+1}\partial_\phi\right) \tag{24}$$

and satisfying $\Box\xi^{(n)} = 0$. Repackaging these modes $\xi = \sum_n f_n \xi^{(n)}$ and defining $f(\tau) = \sum_n f_n e^{in\tau}$, one finds the large η behavior

$$\xi \approx -f(\tau)\partial_\tau + f'(\tau)\partial_\eta + i f(\tau)\partial_\phi. \tag{25}$$

These diffeomorphisms therefore correspond to boundary time reparametrizations that send $\tau \to \tau - f(\tau)$, $\eta \to \eta + f'(\tau)$, and $\phi \to \phi + i f(\tau)$ and resemble vector fields appearing in Kerr/CFT [34–37]. The path integral (20) is therefore proportional to an integral over the (infinite-dimensional, non-compact) coset $\mathrm{Diff}(S^1)/SL(2,\mathbb{R})$. The quotient by $SL(2,\mathbb{R})$ arises because the $n = 0, \pm 1$ perturbations, which would correspond to diffeomorphisms generated by (17) (i.e., $\mathcal{L}_{L_{\pm1},L_0}\bar{g}$) vanish due to the isometries of the background metric. This symmetry breaking pattern, explicated in [20], is known to control many aspects of the near-extremal thermodynamics of spherically symmetric black holes. Since the mode (25) costs no action and has infinite volume, the one-loop approximation to the path integral suffers from an infrared divergence

$$Z \propto \int_{\mathrm{Diff}(S^1)/SL(2,\mathbb{R})} [Df(\tau)] = \infty, \tag{26}$$

which is totally independent of any UV completion and completely controlled by the low energy fields in the model.

The not-NHEK metric: The infinity (26) is an infrared divergence, which arises from low energy modes of low energy fields, and is therefore a physical effect. Its existence calls into question the basic assumption that the NHEK path integral computes the zero-temperature black hole partition function. One way to settle this question would be to define and compute the finite temperature partition function for the black hole and then to take the $T \to 0$ limit. In other words, we would like to know if $\lim_{T\to0} Z[T]_{\text{Black Hole}}$ reproduces $(Z_{\rm NHEK})_{\rm reg}$, a properly regulated version of the throat partition function with quantum fluctuations taken into account.

The issue is that "$Z[T]_{\text{Black Hole}}$" is itself difficult to define, let alone compute, at finite temperature in asymptotically flat space. The most obvious definition would involve a Euclidean path integral with the standard asymptotically flat boundary conditions and a periodic identification of asymptotic Euclidean time. This calculation cannot be performed explicitly for the near-extreme Kerr black hole beyond the leading saddle-point approximation.

The authors of [7,19] adopt a different definition of $Z[T]_{\text{Black Hole}}$ for the low temperature Reissner-Nordström black hole, and their main conclusion is that in the absence of supersymmetry, $(Z_{AdS_2\times S^2})_{\rm reg} \equiv \lim_{T\to0} Z[T]_{\text{Black Hole}} \neq e^{S_0}$. We review these calculations in the Supplemental Material [33], Appendix A, which includes Refs. [38,39]. We extend the analysis to the Kerr black hole below.

The main assumption underlying their definition of $Z[T]_{\text{Black Hole}}$ is that, for small temperatures, one can simply correct the throat geometry (18) rather than perform the full asymptotically flat path integral. It is not obvious that this is mathematically equivalent to taking the small T limit of the full path integral with asymptotically flat boundary conditions, but it seems physically plausible that the leading corrections to the low temperature thermodynamics arise from dynamics near the throat. Either way, since the full finite temperature black hole certainly does not support the infinite set of zero modes (23), it is clear that the IR divergence will disappear in either prescription. Whether the form of the correction is the same is less obvious.

At a technical level, the prescription amounts to performing the diffeomorphism (15), and then expanding the resulting metric in powers of T instead of taking the strict $T \to 0$ limit. The leading term is of course the NHEK metric (18). The subleading term represents a (non-normalizable) gravitational perturbation of NHEK whose nonlinear completion is the asymptotically flat finite temperature Kerr black hole, as in [23,24]. The $O(T)$ correction to the Wick-rotated metric [denoted by δg in (4)] in our conventions is given by

$$\begin{aligned}\frac{\delta g_{\mu\nu}dx^\mu dx^\nu}{4\pi J^{3/2}T} &= (1+\cos^2\theta)(2+\cosh\eta)\tanh^2\frac{\eta}{2}(d\eta^2 - \sinh^2\eta d\tau^2) + \sin^2\theta\cosh\eta(d\eta^2+\sinh^2\eta d\tau^2) + 2\cosh\eta d\theta^2 \\ &\quad + 2\frac{\sin^2\theta}{1+\cos^2\theta}(\cosh\eta-1)\left((\sin^2\theta\sinh^2\eta-3) - 4\frac{\cos^2\theta}{1+\cos^2\theta}\cosh\eta(\cosh\eta-1)\right)d\tau^2 \\ &\quad + 2i\frac{\sin^2\theta}{1+\cos^2\theta}\left((\sin^2\theta\sinh^2\eta-3) - 8\frac{\cos^2\theta}{1+\cos^2\theta}\cosh\eta(\cosh\eta-1)\right)d\tau d\phi + 8\cosh\eta\frac{\sin^2\theta\cos^2\theta}{(1+\cos^2\theta)^2}d\phi^2.\end{aligned} \tag{27}$$

Importantly, $\mathcal{L}_{L_0}\delta g = 0$ while $\mathcal{L}_{L_\pm}\delta g = \mathbf{g}_\pm$ with $\mathbf{g}_\pm$ nonnormalizable, so that one is still not integrating over the $n = 0, \pm 1$ modes that would correspond to $SL(2, \mathbb{R})$ diffeomorphisms. As noted in (6), the NHEK zero modes (23) do not result from diffeomorphisms of this corrected geometry, so we expect their eigenvalues to pick up corrections of order T.

Eigenvalue corrections to the extremal zero modes and $\log T$ corrections to the entropy: The correction (27) to the NHEK metric induces a correction δD to the NHEK kinetic operator $\bar{D}$ in (22). This in turn modifies the extremal eigenfunctions h^0 and their eigenvalues Λ^0. Expanding everything to first order in T,

$$(\bar{D} + \delta D)(h_n^0 + \delta h_n) = (\Lambda_n^0 + \delta\Lambda_n)(h_n^0 + \delta h_n), \quad (28)$$

and isolating the $O(T)$ terms, we get

$$\bar{D}\delta h_n + \delta D h_n^0 = \Lambda_n^0 \delta h_n + \delta\Lambda_n h_n^0. \quad (29)$$

Taking the inner product with h_m^0, using orthonormality of the 0th order eigenfunctions, and restoring indices, the 1st order correction to the eigenvalue takes the form

$$\delta\Lambda_n = \int d^4x\sqrt{\bar{g}}(h_n^0)_{\alpha\beta}\delta D^{\alpha\beta,\mu\nu}(h_n^0)_{\mu\nu}. \quad (30)$$

The corrected one loop determinant is therefore

$$\log Z = -\frac{1}{2}\sum_n \log(\Lambda_n^0 + \delta\Lambda_n), \quad (31)$$

with $\delta\Lambda_n \sim T$. This makes it clear that modes that have nonzero extremal eigenvalues ($\Lambda_n^0 \neq 0$) produce subleading, polynomially suppressed T dependence relative to the modes whose extremal eigenvalues vanish. The latter are precisely the real and imaginary parts of (23). The leading order correction to the kinetic operator is

$$\delta D^{\alpha\beta,\mu\nu} = -\frac{1}{16\pi}\delta\left(\frac{1}{4}g^{\alpha\mu}g^{\beta\nu}\Box - \frac{1}{8}g^{\mu\nu}g^{\alpha\beta}\Box + \frac{1}{2}R^{\alpha\mu\beta\nu}\right) \quad (32)$$

with $g_{\text{not-NHEK}} = \bar{g} + \delta g$. The operator itself is utterly intractable, but the quantity (30) with h_n^0 given by the real and imaginary parts of (23) simplifies dramatically and takes the form

$$\delta\Lambda_n = \frac{3nT}{64J^{1/2}}, \qquad n \geq 2. \quad (33)$$

As an aid to readers we record the following intermediate result, where the first term is the Riemann contribution and the second term comes from the Laplacian:

$$\int d^4x\sqrt{\bar{g}}(h_n^0)_{\alpha\beta}\delta D^{\alpha\beta,\mu\nu}(h_n^0)_{\mu\nu} = -\frac{3n(n^2-1)T}{128J^{1/2}}\int_0^\infty d\eta\left[16(\pi-2)\coth\eta\,\text{csch}^2\eta\tanh^{2n}\left(\frac{\eta}{2}\right) - \text{csch}^3\eta\,\text{sech}^4\left(\frac{\eta}{2}\right)\right.$$
$$\times((\pi-2)\cosh 3\eta + [4(n-2)n + 7\pi - 30]\cosh\eta - 2(n - 2\pi + 4)\cosh 2\eta$$
$$\left. + 2n(4n+7) + 4\pi)\tanh^{2n}\left(\frac{\eta}{2}\right)\right]. \quad (34)$$

The contribution of the extremal zero modes to the not-NHEK partition function is therefore

$$\delta\log Z = 2\cdot(-1/2)\sum_{n\geq 2}\log\delta\Lambda_n = \log\left(\prod_{n\geq 2}\frac{64J^{1/2}}{3nT}\right), \quad (35)$$

where the factor of 2 comes from including the identical contributions from the real and imaginary parts of the perturbations. Using zeta function regularization to compute the infinite product

$$\prod_{n\geq 2}\frac{\alpha}{n} = \frac{1}{\sqrt{2\pi}}\frac{1}{\alpha^{3/2}}, \quad (36)$$

the final answer takes the form

$$\delta\log Z = \log\left(\frac{\sqrt{27}}{512\sqrt{2\pi}}\frac{T^{3/2}}{J^{3/4}}\right) \sim \frac{3}{2}\log T. \quad (37)$$

We conclude that at low temperatures

$$Z[T]_{\text{Black Hole}} \sim T^{3/2}e^{S_0} + \text{higher order terms}. \quad (38)$$

It remains to understand the regime of validity of this expression and its physical content. Obviously, once the small T-dependent prefactor begins competing with the large temperature independent exponential, the approximation is not valid. This occurs when $T^{3/2} \sim e^{-S_0}$. Below this temperature, the partition function is so small that other saddles will begin competing with the computation performed here. Similarly, when $T \sim J^{-1/2}$ the linear term in (13) competes with the leading S_0 term and the near-extremal approximation breaks down. Equivalently, the correction term (27) becomes as large as the NHEK metric (16) throughout the throat and the small-T approximation of the geometry completely breaks down.

The fact that the partition function vanishes as $T \to 0$ means that $\rho(E) \to 0$ as $E \to 0$. There is no exponential ground state degeneracy or thermodynamic mass gap [1]. Rather, the would-be ground states are spread out over a dense energy band above the vacuum. Hence, standard thermodynamics still applies in the range $J^{a_1} e^{-a_2 S_0} \lesssim T \lesssim J^{-1/2}$, where a_1, a_2 are expected to be $O(1)$ numbers.

This interpretation is subtle because spontaneous emission of superradiant particles from a zero temperature Kerr black hole serves to spin down the black hole [40]. Once lifted from extremality, the normal Hawking evaporation process takes over, so the lifetime for an extremal Kerr black hole is not qualitatively different from that of Schwarzschild. This effect is partially observable within the NHEK geometry, which supports superradiant modes (not treated in this Letter) which are oscillatory near the boundary and which carry away energy and angular momentum [26].

Although the decay rate can likely be estimated using these modes in the throat, the precise coefficient is not calculable since the ergosphere extends far from the mouth of the throat. Calculating the widths of the Kerr microstates therefore likely requires the full Kerr geometry, and we do not address this aspect of the spectrum in this Letter. A proper mathematical description of the microstates of a Kerr black hole connected to an asymptotically flat region really involves a density of resonances in the complex energy plane, and we can ask about the distribution of energies in the real and imaginary directions. While the widths of the states with $J \sim M^2$ require calculations in the full Kerr geometry, the distribution along the real axis near extremality is controlled by the throat calculation performed in this Letter.

Note added.—Recently, we became aware that similar results are in preparation by Rakic, Rangamani, and Turiaci [25].

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[1] J. Preskill, P. Schwarz, A. D. Shapere, S. Trivedi, and F. Wilczek, Limitations on the statistical description of black holes, Mod. Phys. Lett. A **06**, 2353 (1991).

[2] J. M. Maldacena and A. Strominger, Universal low-energy dynamics for rotating black holes, Phys. Rev. D **56**, 4975 (1997).

[3] L. V. Iliesiu and G. J. Turiaci, The statistical mechanics of near-extremal black holes, J. High Energy Phys. 05 (2021) 145.

[4] M. Heydeman, L. V. Iliesiu, G. J. Turiaci, and W. Zhao, The statistical mechanics of near-BPS black holes, J. Phys. A **55**, 014004 (2022).

[5] J. Boruch, M. T. Heydeman, L. V. Iliesiu, and G. J. Turiaci, BPS and near-BPS black holes in AdS_5 and their spectrum in $\mathcal{N} = 4$ SYM, arXiv:2203.01331.

[6] L. V. Iliesiu, S. Murthy, and G. J. Turiaci, Black hole microstate counting from the gravitational path integral, arXiv:2209.13602.

[7] L. V. Iliesiu, S. Murthy, and G. J. Turiaci, Revisiting the logarithmic corrections to the black hole entropy, arXiv: 2209.13608.

[8] A. Sen, Revisiting localization for BPS black hole entropy, arXiv:2302.13490.

[9] Obstacles to decoupling, not considered here, may arise from superradiant modes.

[10] A. Sen, Microscopic and macroscopic entropy of extremal black holes in string theory, Gen. Relativ. Gravit. **46**, 1711 (2014).

[11] Related results also appeared in the context of black holes with AdS asymptotics; see, for instance, [12–15].

[12] J. T. Liu, L. Pando Zayas, V. Rathee, and W. Zhao, One-loop test of quantum black holes in anti–de Sitter space, Phys. Rev. Lett. **22**, 221602 (2018).

[13] K. Hristov and V. Reys, Factorization of log-corrections in AdS_4/CFT_3 from supergravity localization, J. High Energy Phys. 12 (2021) 031.

[14] M. David, V. Godet, Z. Liu, and L. Pando Zayas, Logarithmic corrections to the entropy of rotating black holes and black strings in AdS_5, J. High Energy Phys. 04 (2022) 160.

[15] M. David, V. Godet, Z. Liu, and L. Pando Zayas, Non-topological logarithmic corrections in minimal gauged supergravity, J. High Energy Phys. 08 (2022) 043.

[16] J. M. Maldacena, J. Michelson, and A. Strominger, Anti-de Sitter fragmentation, J. High Energy Phys. 02 (1999) 011.

[17] A. J. Amsel, G. T. Horowitz, D. Marolf, and M. M. Roberts, No dynamics in the extremal Kerr throat, J. High Energy Phys. 09 (2009) 044.

[18] A. Sen, Logarithmic corrections to rotating extremal black hole entropy in four and five dimensions, Gen. Relativ. Gravit. **44**, 1947 (2012).

[19] N. Banerjee and M. Saha, Revisiting leading quantum corrections to near extremal black hole thermodynamics, J. High Energy Phys. 07 (2023) 010.

[20] J. Maldacena, D. Stanford, and Z. Yang, Conformal symmetry and its breaking in two dimensional nearly anti-de-Sitter space, Prog. Theor. Exp. Phys. **2016**, 12C104 (2016).

[21] The terminology near-NHEK is already commonly used to denote the leading order geometry (4), which is really a Rindler patch of the full global NHEK geometry. It should not be confused with the term "nearly AdS_2" which describes the Reissner-Nordström analog of the metric (4).

[22] U. Moitra, S. K. Sake, S. P. Trivedi, and V. Vishal, Jackiw-Teitelboim gravity and rotating black holes, J. High Energy Phys. 11 (2019) 047.

[23] A. Castro and V. Godet, Breaking away from the near horizon of extreme Kerr, SciPost Phys. **8**, 089 (2020).

[24] A. Castro, V. Godet, J. Simón, W. Song, and B. Yu, Gravitational perturbations from NHEK to Kerr, J. High Energy Phys. 07 (2021) 218.

[25] I. Rakic, M. Rangamani, and G. J. Turiaci, Near extremal Kerr and its entropy, *ExU-YITP Workshop on Holography, Gravity and Quantum Information* (2023), https://www2.yukawa.kyoto-u.ac.jp/qimg2023/presentation_files/Rangamani_Mukund_09_14_.pdf.

[26] J. M. Bardeen and G. T. Horowitz, The extreme Kerr throat geometry: A vacuum analog of $AdS_2 \times S^2$, Phys. Rev. D **60**, 104030 (1999).

[27] Some references refer to this spacetime as near-NHEK, and reserve the term NHEK for its geodesic completion.

[28] Although embeddings of nonsupersymmetric Kerr-like rotating black holes in string theory can be found in [29,30].

[29] M. Guica and A. Strominger, Microscopic realization of the Kerr/CFT correspondence, J. High Energy Phys. 02 (2011) 010.

[30] G. Compere, W. Song, and A. Virmani, Microscopics of extremal Kerr from spinning M5 branes, J. High Energy Phys. 10 (2011) 087.

[31] S. Bhattacharyya, B. Panda, and A. Sen, Heat kernel expansion and extremal Kerr-Newmann black hole entropy in Einstein-Maxwell theory, J. High Energy Phys. 08 (2012) 084.

[32] The analysis in [18] proceeds by dimensionally reducing NHEK along the angular directions. The gauge field arising from the ∂_ϕ isometry is then argued to furnish another set of vector zero modes analogous to the AdS_2 perturbations in Eq. (16) of Appendix A [33], although no explicit formula is presented. The naive guess corresponding to $\xi^{(n)} \propto \Phi_n(\tau, \eta)\partial_\phi$ with Φ_n defined in Eq. (17) of Appendix A [33] does not satisfy the harmonic gauge condition and is not a zero mode of the operator (22). Note that $\xi^{(n)} \propto \Phi_n(\tau, \eta)\partial_\phi$ already appears as part of the diffeomorphism (24) that generates the tensor zero modes (23). A similar twist of conformal transformations by large U(1) gauge transformations in AdS_2 is described in [34]. Regardless, as argued in [7], these vector zero modes are only expected to contribute in a specific choice of ensemble, and so are irrelevant to the universal, ensemble-independent $\frac{3}{2}\log T$ correction that we compute in this Letter.

[33] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.133.021601 for the computation of the logarithmic corrections to the entropy for the extremal Reissner-Nordström black hole.

[34] T. Hartman and A. Strominger, Central charge for AdS_2 quantum gravity, J. High Energy Phys. 04 (2009) 026.

[35] M. Guica, T. Hartman, W. Song, and A. Strominger, The Kerr/CFT correspondence, Phys. Rev. D **80**, 124008 (2009).

[36] A. Castro and F. Larsen, Near extremal Kerr entropy from AdS_2 quantum gravity, J. High Energy Phys. 12 (2009) 037.

[37] I. Bredberg, C. Keeler, V. Lysov, and A. Strominger, Cargese lectures on the Kerr/CFT correspondence, Nucl. Phys. B, Proc. Suppl. **216**, 194 (2011).

[38] A. Sen, Logarithmic corrections to $N = 2$ black hole entropy: An infrared window into the microstates, Gen. Relativ. Gravit. **44**, 1207 (2012).

[39] R. Camporesi and A. Higuchi, Spectral functions and zeta functions in hyperbolic spaces, J. Math. Phys. (N.Y.) **35**, 4217 (1994).

[40] D. N. Page, Particle emission rates from a black hole. 2. Massless particles from a rotating hole, Phys. Rev. D **14**, 3260 (1976).