Successful Freeze-Out of Strongly Interacting Dark Matter with Even-Numbered Interactions

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Strongly interacting massive particles π have been advocated as prominent dark matter candidates when they regulate their relic abundance through odd-numbered $3\pi \to 2\pi$ annihilation. We show that successful freeze-out may also be achieved through *even-numbered* interactions $XX \to \pi\pi$ once bound states X among the particles of the low-energy spectrum exist. In addition, X-formation hosts the potential of also *catalyzing* odd-numbered $3\pi \to 2\pi$ annihilation processes, turning them into effective two-body processes $\pi X \to \pi\pi$. Bound states are often a natural consequence of strongly interacting theories. We calculate the dark matter freeze-out and comment on the cosmic viability and possible extensions. Candidate theories can encompass confining sectors without a mass gap, glueball dark matter, or ϕ^3 and ϕ^4 theories with strong Yukawa or self-interactions.

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Introduction.—The several decades-long efforts to detect dark matter (DM) nongravitationally have, to a significant degree, been fueled by a relic density argument: the couplings of DM to the standard model (SM) that allow for phenomenological exploration also successfully generate the DM abundance in the early Universe through thermal two-body annihilation of DM into SM states. As has been realized more recently, when the link between DM and SM becomes too weak, DM may still regulate its abundance through $3 \to 2$ DM-only processes [1]; also [2–4]. This offers a pathway of DM as a thermal relic even when being partially secluded from the SM. The odd-numbered $3 \to 2$ reaction is naturally realized through the Wess-Zumino-Witten (WZW) five-point interaction of a strongly interacting dark sector [5]. The possibility of a DM number-depleting process that proceeds without participation of additional degrees of freedom is hence very attractive and has led to a flurry of further exploration, see Refs. [6–28] among others.

In this Letter, we show that bound states $X \equiv [\pi\pi]$ among strongly interacting massive particles (SIMPs), denoted as π, impact relic abundance predictions, potentially altering the conventional understanding of the "SIMP mechanism." In particular, X may lead to a catalysis of freeze-out reactions by adding new channels

$$\text{catalyzed } 3 \to 2 \text{ annihilation:}\ \pi + X \to \pi + \pi, \tag{1a}$$

$$\text{catalyzed } 4 \to 2 \text{ annihilation:}\ X + X \to \pi + \pi. \tag{1b}$$

The last two are effective $2 \to 2$ processes and compete with the free $3\pi \to 2\pi$ and $4\pi \to 2\pi$ counterpart reactions in depleting the overall DM mass density. Moreover, whereas $3\pi \to 2\pi$ and (1a) are related through the same underlying odd-numbered interaction, the final process $XX \to \pi\pi$ can be entirely due to *even-numbered* interactions, such as the four-point self-interaction. This releases a requirement on the interaction structure of the theory and opens the door to a SIMP mechanism without relying on anomaly-mediated interactions.

Of course, the prospect of catalyzed reactions (1a) and (1b) taking place requires X to be part of the low-energy spectrum. Once the theory allows for the existence of sufficiently long-lived X, their formation is guaranteed through the radiationless exoergic process

$$\text{guaranteed } X \text{ formation:}\ \pi + \pi + \pi \to \pi + X. \tag{2}$$

This reaction may also be mediated through even-numbered interactions, and in its effective strength, it is not suppressed relative to the standard $3 \to 2$ process. Hence, X may form efficiently, and it shows that already for models of SIMPs in isolation, the role of bound states calls to be studied; see Fig. 1 for illustration.

Exemplary SIMP model.—To allow for a paralleling exposition close to the original papers on the SIMP-mechanism [1,5], we shall consider the low-energy

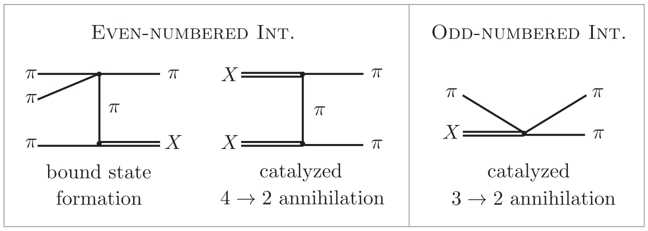

FIG. 1. X-formation; X-assisted annihilation via two 4-point interactions; catalyzed $3 \to 2$ annihilation.

effective theory of $a = 1, \ldots, N_\pi$ massive pseudo-Goldstone bosons π_a as the DM candidates emerging from a confining dark non-Abelian gauge group of N_f fermion fields. The dynamics discussed is not exclusive to this choice and further possibilities will be commented on.

In the construction of the chiral Lagrangian, π_a are written as fluctuations of the orientation of the chiral condensate Σ_0, $\Sigma = e^{i\pi/f_\pi}\Sigma_0 e^{i\pi^T/f_\pi}$, with $\pi = \sum_n \pi_n T^n$ where T^n are the N_π broken generators of the flavor group with normalization $\mathrm{Tr}[T^a T^b] = \delta^{ab}/2$. Expanding in terms of Σ yields the canonically normalized kinetic terms, masses, and even-numbered interactions of π. Considering a flavor-degenerate quark mass matrix M with entries m, their universal mass is given by $m_\pi^2 = \pm 2\mu^3 m/f_\pi^2$; f_π is the decay constant, and the plus (minus) sign applies to $\mathrm{Sp}(2N_f)$ [$\mathrm{SU}(N_f)$ or $\mathrm{SO}(N_f)$] residual flavor symmetry. Interactions are given by

$$\mathcal{L}_{\mathrm{int}}^{\mathrm{even}} \supset -\frac{1}{3f_\pi^2}\mathrm{Tr}([\pi, \partial_\mu \pi][\pi, \partial^\mu \pi]) + \frac{m_\pi^2}{3f_\pi^2}\mathrm{Tr}[\pi^4] \quad (3)$$

plus higher order terms $O(\pi^6/f_\pi^6)$. Odd-numbered interactions in form of a nonvanishing WZW term are only present for symmetry-breaking pattern with coset spaces with nontrivial fifth homotopy groups [29]. The leading order WZW Lagrangian then reads,

$$\mathcal{L}_{\mathrm{int}}^{\mathrm{odd}} = \frac{2N_c}{15\pi^2 f_\pi^5}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}[\pi\partial_\mu\pi\partial_\nu\pi\partial_\rho\pi\partial_\sigma\pi]. \quad (4)$$

In the picture of strongly interacting theories, X would be a "meson molecule" or "tetraquark" of mass $m_X = 2m_\pi - E_B$ and $E_B > 0$. For the exposition of our ideas, we assume a shallow bound molecule with $\kappa \equiv E_B/m_\pi \sim 0.1$ so that it can be treated as a nonrelativistic bound state [30]. This points to a theory with $m/\Lambda \lesssim 1$ where $\Lambda \simeq 2\pi f_\pi$ [31]. Theories with $m/\Lambda \ll 1$ such as for light quarks in QCD have a mass-gap whereas for $m/\Lambda \gg 1$, the lowest lying states are expected to be gluonia. The general ideas presented here also apply to deeper bound systems, but their treatment requires advanced field theoretical tools, greatly complicating matters. Even if we are far from the chiral limit, using (3) and (4) allows for a most direct comparison with the original SIMP idea. In the following, we shall take an $Sp(2N_c)$ gauge theory ($N_c = 2$) for concreteness with $N_f = 2$ fundamental Dirac fermions. After chiral symmetry breaking, the vacuum alignment is $\Sigma_0 = E$, where E is the unitized invariant matrix of the remaining $Sp(4)$ flavor symmetry group. A detailed study of this choice is presented in [32] (see also previous lattice studies [33,34]) and there are works studying the existence of X in such scenarios, e.g., [35–38]. Another strongly interacting option is taking $m/\Lambda \gg 1$ with glueball DM [13,39–41] and to consider their bound states [42]. This is left for future work [43].

Bound state-assisted SIMP annihilation.—Before a detailed analysis, simple estimates may convince us that bound state formation and X-assisted annihilation are both efficient and may even supersede odd-numbered interactions. The parametric ratio of rates of X formation to odd-numbered annihilation reads,

$$\frac{\Gamma_{3\pi\to\pi X}}{\Gamma_{3\pi\to 2\pi}} = \frac{\langle\sigma_{3\pi\to\pi X}v^2\rangle}{\langle\sigma_{3\pi\to 2\pi}v^2\rangle} \approx \frac{|\psi(0)|^2 f_\pi^2}{m_\pi^5}x_\mathrm{f}^2. \quad (5)$$

Here $\langle\sigma_i v^2\rangle$ are the thermally averaged collision integrals of the respective processes. The dimensionful factor that relates both "cross sections" is the square of the bound state wave function at the origin $|\psi(0)|^2$; $x_\mathrm{f}^2 \equiv (m_\pi/T_\mathrm{f})^2 \sim 400$ is an enhancement factor that accounts for the different velocity scalings of rates, D wave for $3\pi \to 2\pi$ and S wave for $3\pi \to \pi X$, at nonrelativistic freeze-out temperature $T_\mathrm{f} \sim m_\pi/20$. Taking as an estimate $|\psi(0)|^2 = 1/a_B^3$ with the Bohr radius $a_B = 2/(\alpha_s' m_\pi)$ given in terms of the dark strong coupling constant $\alpha_s' = O(1)$ shows that the ratio in (5) easily exceeds unity on account of $m_\pi/f_\pi = O(1)$ in strongly interacting theories [44].

We may also convince ourselves that X-assisted annihilation competes with its free counterpart. Naive dimensional analysis suggests that, for a pair of π particles, it is more likely for them to meet as constituents of a bound state than as free particles,

$$\frac{n_X|\psi(0)|^2}{n_\pi^2} \approx 2\sqrt{2}\pi^{3/2}x_\mathrm{f}^{3/2}e^{\kappa x_\mathrm{f}}\frac{|\psi(0)|^2}{m_\pi^3}, \quad (6)$$

where we used nonrelativistic Maxwell-Boltzmann statistics for n_π and n_X. Note that $2\sqrt{2}\pi^{3/2}x_\mathrm{f}^{3/2} \approx 10^3$ for $x_\mathrm{f} = 20$. Thus, the ratio in (6) may easily exceed unity and suggests on general grounds that $XX \to \pi\pi$ dominates over $4\pi \to 2\pi$, and that $\pi X \to \pi\pi$ dominates over $3\pi \to 2\pi$ when odd-numbered interactions are present. This is what we mean by "catalysis."

Cross sections.—We now calculate the relevant cross sections from (3) and (4) and first consider the X-formation cross section $\langle\sigma_{3\pi\to\pi X}v^2\rangle$ [46]. In the approximation that we are working in, the amplitude for a bound state process is obtained from the free amplitude as follows. Following the notation of [47] and calling $\langle\vec{k}_1,\vec{k}_2,\vec{k}_3|\mathcal{T}\{\ldots\}|\vec{p}_1,\vec{p}_2,\vec{p}_3\rangle$ the matrix-element of $3\pi\to 3\pi$ with respective incoming and outgoing momenta $\vec{p}_i$ and $\vec{k}_i$, the amplitude for the bound state formation $3\pi\to\pi X$ is obtained from

$$\frac{\sqrt{2M_X}}{2m_\pi}\int\frac{d^3\vec{q}}{(2\pi)^3}\left\langle\frac{\vec{K}}{2}-\vec{q},\frac{\vec{K}}{2}+\vec{q},\vec{k}_3|\mathcal{T}\{\ldots\}|\vec{p}_1,\vec{p}_2,\vec{p}_3\right\rangle\tilde{\psi}(\vec{q}). \tag{7}$$

Here, $\vec{K}$ is the three-momentum of X; $\tilde{\psi}(\vec{q})$ is the Fourier transform $\tilde{\psi}(\vec{q})=\int d^3\vec{x}\psi(\vec{x})e^{-i\vec{q}\cdot\vec{x}}$ of the wave function that is a solution to the nonrelativistic Schrödinger equation with confining potential $V(r)$, where $r=|\vec{x}|$ is the separation of the constituent SIMPs.

On general grounds, many terms contribute to the integral in (7). They can be classified by the power of the Cartesian components of q_i that enter through the matrix element, enforcing a selection rule on $\psi(\vec{x})$. Decomposing the latter into angular and radial parts, $\psi(\vec{x})=Y_{\ell m}(\hat{x})R(r)$, constant terms $(q_i)^0$ yield for the integral $\psi(\vec{x}=0)$ which is only nonvanishing for S states $\ell=0$ where $R(0)=\sqrt{4\pi}|\psi(0)|\neq 0$. This is our most important case, as it concerns the ground state of X. Terms proportional to q_i yield the derivative of the radial wave function at the origin, $R'(0)\equiv dR/dr|_{r=0}$, so the lowest contributing angular momentum state is P wave with $\ell=1$. We will encounter this case for the WZW interaction below. Finally, terms quadratic in q_i yield nominally divergent integrals in (7), probing the short distance behavior of the theory. In dimensional regularization one may show that $\nabla^2R(0)=-m_\pi E_B R(0)$ holds [50,51]. Since we take $E_B\ll m_\pi$, this becomes subleading in processes involving S states, and we are hence allowed to neglect such contributions.

For the bound state formation process $3\pi\to\pi X$, there are then various diagrams to consider. During nonrelativistic freeze-out, the dominant processes are t- and u-channel type diagrams of the sort depicted in Fig. 1 where two 4-point interactions are connected via a SIMP propagator. The denominators of the propagators are enhanced by a matching kinematic condition $k^2-m_\pi^2\propto -E_B m_\pi$. This renders other diagrams irrelevant. The cross section is then given by

$$\langle\sigma_{3\pi\to\pi X}v^2\rangle\simeq\frac{57\,041}{1\,310\,720\sqrt{3}\pi^2}\frac{R_S^2(0)}{f_\pi^8}\left(\frac{m_\pi}{E_B}\right)^{3/2}. \tag{8}$$

The prefactor depends on the gauge group and symmetry breaking pattern and is averaged all possible incoming and summed over all outgoing flavor combinations so that $n_\pi^3\langle\sigma_{3\pi\to\pi X}v^2\rangle$ yields the total number change per time. In obtaining the result, we used a nonrelativistic expansion, assuming that the typical incoming kinetic energy satisfies $m_\pi\langle v^2\rangle/2\lesssim E_B$. This is equivalent to demanding $T_\mathrm{f}\lesssim E_B$. The thermal average over a Maxwell-Boltzmann ensemble was taken in the final step. When comparing with (5) we observe an additional enhancement by a factor of $(m_\pi/E_B)^{3/2}$ in the ratio of rates relative to the WZW-mediated $3\pi\to 2\pi$ annihilation. Detailed calculations of all cross sections are provided in the Supplemental Material [46] to this Letter.

Similarly, we may proceed with the calculation of the annihilation cross section for $XX\to\pi\pi$. A general computation would be a formidable challenge, but we may again profit from the imposed selection rules, focusing on S-wave initial bound states. There are six t- and u-channel diagrams, which become related in the limit that we neglect the internal motion of the constituents. The final result reads

$$\langle\sigma_{XX\to\pi\pi}v\rangle\simeq\frac{2\,529\,757}{424\,673\,280\sqrt{3}\pi^3}\frac{R_S^4(0)}{f_\pi^8}, \tag{9}$$

where we have again summed over all flavor combinations. Finally, when considering odd-numbered $3\to 2$ interactions enabled by (4), we obtain the cross section for the related $\pi X\to\pi\pi$ process as

$$\langle\sigma_{\pi X\to\pi\pi}v\rangle\simeq\frac{\sqrt{5}N_c^2m_\pi^3}{512\pi^6f_\pi^{10}x}R_P'^2(0). \tag{10}$$

Importantly, the process requires X to be in a P-wave state X_P with $\ell=1$.

Abundance evolution.—We are now in a position to solve the evolution equations for the two populations, free π and X. Their respective total comoving number densities, normalized to the total entropy density s, are given by $Y_{\pi,X}=n_{\pi,X}/s$, where s is the total entropy density of the Universe and a sum over all flavors is implicit. We assume that kinetic equilibrium with SM is maintained; we develop this in the next section. At high temperatures (small x), π and X follow their equilibrium distributions, $Y_{\pi,X}=Y_{\pi,X}^{\mathrm{eq}}$, due to fast number-changing processes. Their chemical decoupling happens at $x_1\sim 20$ when $n_X^2\langle\sigma_{XX\to\pi\pi}v\rangle/n_\pi\simeq H(x_1)$. Subsequently, considering only even-numbered interactions, assuming dominance of $XX\to\pi\pi$ over the free $4\pi\to 2\pi$ counterpart and neglecting the inverse process, a particularly simple form of the Boltzmann equation is found for the combination $Y_\pi+2Y_X$ [52],

$$\frac{d(Y_\pi+2Y_X)}{dx}=-\frac{2\langle\sigma_{XX\to\pi\pi}v\rangle Y_X^2s(x)}{xH(x)}\quad(x>x_1). \tag{11}$$

The immediate evolution that ensues for $x>x_1$ is nontrivial because bound state formation $3\pi\leftrightarrow\pi X$ is still operative. This maintains a detailed balance between the X and π populations,

$$Y_X = \frac{Y_\pi^2 Y_X^{\rm eq}}{(Y_\pi^{\rm eq})^2} = Y_\pi^2 \frac{N_X}{N_\pi^2} (2\pi x)^{3/2} e^{\kappa x} \left(\frac{m_X}{m_\pi}\right)^{3/2} \frac{s(x)}{m_\pi^3}, \qquad (12)$$

where $N_\pi = 5$ and $N_X = (N_\pi + 1)N_\pi/2 = 15$ are the possible flavor combinations. Together, (12) and (11) determine the evolution of Y_π and Y_X for as long as their detailed balance holds until bound state formation freezes out at $x_2 > x_1$ when $\Gamma_{3\pi\to\pi X} \equiv n_\pi^2 \langle \sigma_{3\pi\to\pi X} v^2\rangle = H(x_2)$.

Using (12) in (11) with $Y_X \ll Y_\pi$ and neglecting dY_X/dx, the Boltzmann equation becomes one for dY_π/dx that can be integrated. To leading order in x_1/x_2 we obtain the following solution,

$$\begin{aligned} Y_\pi^{-3}(x_2) \simeq{}& \frac{256\sqrt{2}\pi^8 g_*^{5/2} m_\pi M_P \langle \sigma_{XX\to\pi\pi} v\rangle}{6075\sqrt{5} x_2^4} \frac{N_X^2}{N_\pi^4} \\ &\times [8(\kappa x_2)^4 \mathrm{Ei}(2\kappa x_2) \\ &- e^{2\kappa x_2}(3 + 2\kappa x_2 + 2\kappa^2 x_2^2 + 4\kappa^3 x_2^3)], \end{aligned} \qquad (13)$$

where Ei is the exponential integral function and M_P is the reduced Planck mass and g_* are the effective degrees of freedom at x_2. This approximation works for $\kappa x_1 \gtrsim 1$, which is congruent with assuming $E_B > T$ at chemical decoupling. In writing the solution, we have also taken $Y_\pi^{-3}(x_1) \ll Y_\pi^{-3}(x_2)$. To within a factor of 2, we may then put the solution in suggestive form,

$$\Omega_\pi^{\rm even} \sim 0.2 \left(\frac{200\,\kappa x_2^5}{e^{2\kappa x_2}} \frac{\text{bn/GeV}}{\langle \sigma_{XX\to\pi\pi} v\rangle / m_\pi} \frac{m_\pi}{\text{GeV}} \right)^{1/3}. \qquad (14)$$

This is a central result. First, note that the relic density depends on x_2, i.e., the moment of freeze-out of bound state formation, and not x_1. Second, we observe a strong dependence on κ, and with x_2 typically between 50 to 100, $\kappa = 0.1$ suggests sub-GeV DM with cm^2/gram self-interactions—in the same ballpark as the odd-numbered SIMP case. Finally, when compared to ordinary $2 \to 2$ freeze-out, the inverse dependence on the annihilation cross section is softened by the cubic root.

The top panel of Fig. 2 shows the full numerical solution for Y_π and Y_X as a function of x for the parameters given in the caption. The evolution contains three steps as discussed above. First, chemical decoupling happens at $x_1 \simeq 20$ when $XX \leftrightarrow \pi\pi$ freezes out. Both X and π develop a chemical potential and keep a detailed balance via $3\pi \leftrightarrow \pi X$ until $x_2 \simeq 115$. For $x > x_2$ free π have reached their relic density value while Y_X keeps decreasing further, as the rate of $XX \to \pi\pi$ with respect to n_X is still larger than the Hubble rate, $n_X \langle \sigma_{XX\to\pi\pi} v\rangle > H(x_2)$. The evolution of Y_X is shown in the bottom panel. Figure 3 shows that with bound states, even-numbered interactions can achieve the observed DM abundance with bn/GeV-scale self-scattering and reach $m_\pi >$ GeV with perturbative couplings [53].

Odd-numbered case.—We now turn our attention to the scenario when we are additionally afforded odd-numbered

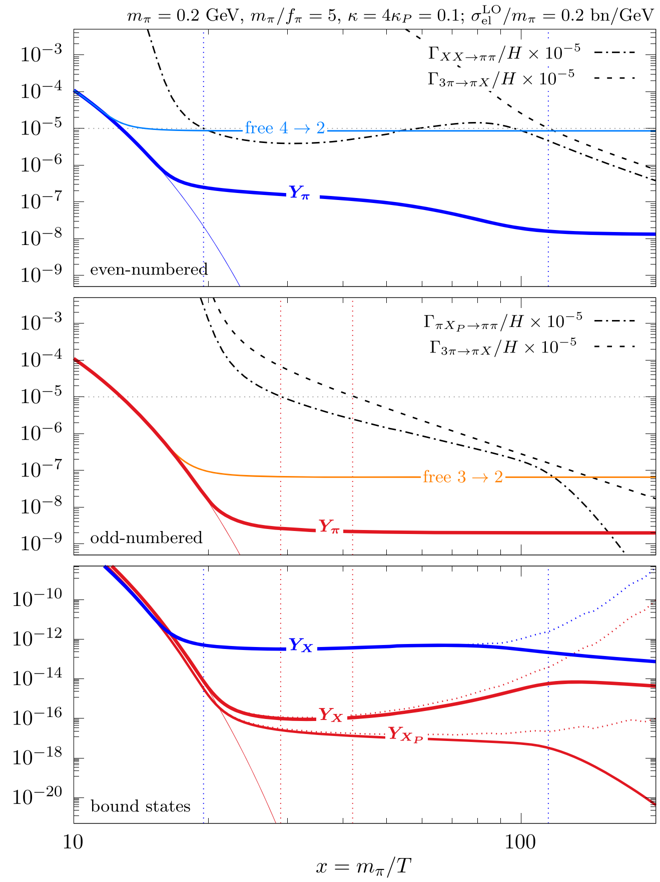

FIG. 2. Evolution of the DM abundance $Y_\pi + 2Y_X \simeq Y_\pi$ for $m_\pi = 0.2$ GeV, $m_\pi/f_\pi = 5$, $\kappa_S = 4\kappa_P = 0.1$, $R(0) = 0.3 m_\pi^{3/2}$, and $dR(0)/dr = 0.06 m_\pi^{5/2}$. Thin solid and dotted lines show the equilibrium and detailed balance abundances. *Top:* even-numbered case where XX annihilation freezes out at $x_1 \simeq 20$ (left vertical line) and bound-state formation freezes out at $x_2 \simeq 115$ (right vertical line); corresponding rates normalized to H are shown as labeled. *Middle:* odd-numbered WZW interactions included. X_P annihilation freezes out comparatively later (left vertical line), maintaining longer chemical equilibrium. Free $4 \to 2$ and $3 \to 2$ freeze-out are shown for comparison in the top and middle panel. *Bottom:* associated bound state abundances with matching colors. For the choice of parameters, the DM abundance is reached for the odd case; using instead $R(0) = 1.4 m_\pi^{3/2}$ yields the DM abundance for the even case.

interactions. As calculated in (10), the efficiency of $\pi X_P \to \pi\pi$ entirely hinges on the availability X_P, which must be present in the low energy spectrum. Since the path to collisional excitation is open, one may consider the detailed balancing relation $n_{X_P}/n_{X_S} = 3\exp[-|\kappa - \kappa_P|x]$ as an estimate for the number density n_{X_P} of excited states; $\kappa_P = E_P/m_\pi$ with E_P being the P-wave binding energy. As we shall see now, the impact of bound states X can also be substantial. Even with the bottleneck of P-wave states for WZW interactions, $\pi X_P \to \pi\pi$ supersede the free $3\pi \to 2\pi$ scenario, and is generally stronger than $XX \to \pi\pi$.

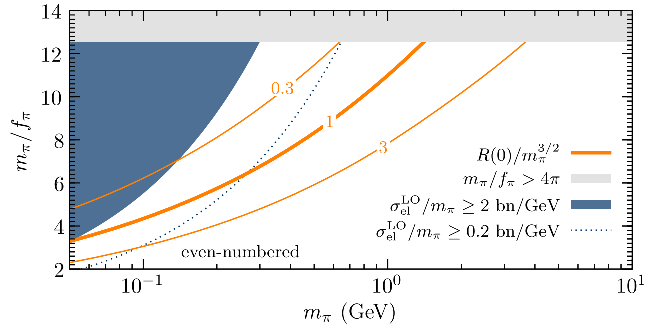

FIG. 3. Contours of observed DM abundance in the even-numbered case for $\kappa = 0.1$ and $R(0)/m_\pi^{3/2} = 0.3$, 1, 3, respectively. The colored region (dotted line) shows the DM self-scattering constraint $\sigma_{\rm el}^{\rm LO} \geq 2\,(0.2)$ bn/GeV. The free $4\pi \to 2\pi$ counterpart would require $m_\pi/f_\pi > 4\pi$ for $m_\pi > 1$ MeV.

For the odd-numbered case, the chemical decoupling happens when $n_{X_P}\langle\sigma_{\pi X_P\to\pi\pi}v\rangle \simeq H(x_1)$. For $x \geq x_1$, the right-hand side of (11) is replaced by $-s\langle\sigma_{\pi X_P\to\pi\pi}v\rangle Y_\pi Y_{X_P}/(xH)$. Since Y_{X_P}/Y_{X_S}, and thus the rate of $\pi X_P \to \pi\pi$, decrease exponentially for $x > x_1$, Y_π already freezes out at x_1. If $3\pi \to \pi X$ decouples later, $x_2 > x_1$, (12) allows us to estimate the yield from $n_{X_P}^{\rm eq}\langle\sigma_{\pi X_P\to\pi\pi}v\rangle/H(x_1) \simeq (Y_\pi^{\rm eq})^2/Y_\pi^2(x_1)$. This gives

$$\Omega_\pi^{\rm odd} \simeq 0.2\left(\frac{x_1}{20}\right)^{5/4}\left(\frac{e^{-\kappa_P x_1}10^{-3}\ {\rm bn/GeV}}{\langle\sigma_{\pi X_P\to\pi\pi}v\rangle/m_\pi}\right)^{1/2}. \qquad (15)$$

The middle panel of Fig. 2 shows the full numerical freeze-out solution. In comparison with the even-numbered case, the stronger $\pi X_P \to \pi\pi$ reaction maintains longer chemical equilibrium. At the same time, with p-wave states diminishing more rapidly with temperature, there is no distinct intermediate phase, and freeze-out happens in one step [unless considering large values for $dR(0)/dr$]. Also shown is the standard SIMP scenario through free $3 \to 2$ reactions, and we observe an order of magnitude smaller freeze-out yield for the chosen set of parameters when X is considered (catalysis). This softens the notorious tension between maximum permissible elastic scattering cross section and relic density requirement in SIMP models [54], and for the shown case, both requirements are indeed satisfied. The leading order elastic scattering cross section is $\sigma_{\rm el}^{\rm LO}/m_\pi = 0.2$ bn/GeV, receiving higher order corrections in the chiral expansion [54] as well as from the bound state in the spectrum. For the latter, we may estimate an S-wave scattering length of the order $1/(\kappa_S m_\pi^2)^{1/2}$ [55], leading to resonant-induced $\sigma_{\rm el}/m_\pi \sim 1/(\kappa_S m_\pi^3)$. For $m_\pi = 0.2$ GeV and $\kappa_S \sim 0.1$ adopted here, it suggests an elastic scattering in the same ballpark and below bn/GeV.

Within the exemplary scenario of pseudo-Goldstone bosons making X, we may also comment on the influence of additional low-lying states, such as ρ mesons of mass $m_\rho \lesssim 2m_\pi$. Additional annihilation channels become available, such as $3\pi \to \pi\rho^* \to \pi\pi$ [20] or $3\pi \to \pi\rho$ [28]. For the parameter region of interest and unless one considers a finely-tuned resonance region $m_\rho \simeq 2m_\pi$, we find that these processes are generally subleading to the X-mediated ones and we are allowed to neglect them.

Coupling to SM and longevity of X.—As is pertinent to all SIMP scenarios that freeze out through self-depletion, kinetic equilibrium with radiation must be maintained to achieve a *cold* DM scenario. An elastic scattering process $\pi{\rm SM}_i \to \pi{\rm SM}_i$ with rate $\Gamma_{\pi{\rm SM}} = \langle\sigma_{\pi{\rm SM}}c\rangle n_i > H$, which brings SM and dark sector into kinetic equilibrium during freeze-out, generally also enables $\pi\pi \to {\rm SM}_i\overline{\rm SM}_i$ annihilation. The SIMP mechanism then requires $\Gamma_{\rm ann} = n_\pi\langle\sigma_{\rm ann}v\rangle < H$, where $\sigma_{\rm ann}$ is the cross section for $\pi\pi \to {\rm SM}_i\overline{\rm SM}_i$. On the account of $n_i/n_\pi \gg 1$, where n_i is the number density of a relativistic SM species, both conditions are generically satisfied [1]. In the current context, interactions of π with SM may also destabilize X through $X = [\pi\pi] \to {\rm SM}_i\overline{\rm SM}_i$ and we must ensure that for the decay rate $\Gamma_X < H$ holds until after freeze-out.

Assuming $\sigma_{\rm ann}v \simeq$ const. and noting that $|\psi(0)|^2 v$ has units of particle flux, we may estimate the induced decay width of X as $\Gamma_X \sim |\psi(0)|^2(\sigma_{\rm ann}v)$, where $(\sigma_{\rm ann}v)$ is the $\pi\pi \to {\rm SM}_i\overline{\rm SM}_i$ annihilation cross section. The longevity requirement $\Gamma_X/H < 1$ thereby translates into an upper limit on the annihilation cross section,

$$\sigma_{\rm ann}v \lesssim 10^{-3}\ {\rm pb}\,x^{-2}\left(\frac{m_\pi}{100\ {\rm MeV}}\right)^2\frac{{\rm MeV}^3}{|\psi(0)|^2}. \qquad (16)$$

In the simplest cases, such as contact interactions through a heavy mediator, elastic and annihilation cross sections are additionally related and in the same ballpark, $\sigma_{\pi{\rm SM}}c \sim \sigma_{\rm ann}v$. We may then use the bound in (16) to estimate the implied ceiling on the elastic scattering rate,

$$1 \lesssim \frac{\Gamma_{\pi{\rm SM}}}{H} \lesssim \frac{10^6}{x^3}\left(\frac{m_\pi}{100\ {\rm MeV}}\right)^3\frac{{\rm MeV}^3}{|\psi(0)|^2}. \qquad (17)$$

This can be easily satisfied at freeze-out for $|\psi(0)| < m_\pi^{3/2}$. We hence conclude that it is possible to retain kinetic equilibrium while maintaining sufficient longevity of X and paired with sub-Hubble two-body annihilation. Therefore, the model-building requirements for coupling the dark sector to the SM are not escalated compared to the standard SIMP mechanism, and one may use the options already entertained in the original work [1].

Finally, additional X formation and breakup reactions may open when introducing couplings to SM. It is important to note that the detailed balancing condition (12) between X and π—being a Saha equation—remains unaltered. If the new processes dominate over $3\pi \leftrightarrow \pi X$, (12) retains its validity *longer*, x_2 will be larger, and the relic density smaller. Hence, if anything, the introduction of SM interactions harbor the potential to make X-assisted freeze-out even more efficient, adding a level of richness, without jeopardizing the overall picture.

Conclusions.—Bound-state-assisted self-depletion offers a novel approach to DM relic density generation. It supports an even-numbered relic SIMP mechanism and enhances odd-numbered counterparts. Both are realized in strongly interacting theories. A broader study of the many aspects mentioned in this Letter, as well as the exploration of other particle-physics realizations giving rise to X, such as glueballs or strong Yukawa forces, will be the subject of upcoming work [43].

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