Inhomogeneous Kinetic Equation for Mixed Neutrinos: Tracing the Missing Energy

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Flavor-dependent neutrino transport is described by a well-known kinetic equation for occupation-number matrices in flavor space. However, in the context of fast flavor conversion, we identify an unforeseen predicament: the pivotal self-induced exponential growth of small inhomogeneities strongly violates conservation of neutrino-neutrino refractive energy. We prove that it is traded with the huge reservoir of neutrino kinetic energy through gradients of neutrino flavor coherence (the off-diagonal piece of the flavor density matrix) and derive the missing gradient terms. The usual equations remain sufficient to describe flavor evolution, at the cost of renouncing energy conservation, which cannot play any role in explaining the numerically observed final state.

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Introduction.—Flavor evolution in a dense neutrino environment is often described by a kinetic equation for the flavor density matrices $\rho(\mathbf{p},\mathbf{x},t)$, where the diagonal entries are the occupation numbers [1–11]. In the homogeneous case, they are defined as the expectation values of the number-operator matrices $\mathcal{D}_{\alpha\beta}=a^{\dagger}_{\beta,\mathbf{p}}a_{\alpha,\mathbf{p}}$ with the flavor indices α and β and $a^{\dagger}_{\alpha,\mathbf{p}}$ the creation operator for a left-handed neutrino of flavor α with momentum $\mathbf{p}$. In the limit of vanishing neutrino masses and flavor mixing, nontrivial flavor evolution can still arise through classical instabilities that would engender strong flavor correlations [12–16]. Dubbed "fast flavor evolution," this subject has been intensely studied in view of practical consequences for neutrino flavor transport in stellar core collapse and binary neutron star mergers [17–28].

Predicting the consequences of fast conversions is thus a central question. In tackling this problem, a key role must be played by conserved quantities, which are the only exact guide into the final state reached after conversions. One such quantity must be energy. However, we show that the usual equations of motion (EOMs) do not conserve energy. In their often-used form, they are

$$(\partial_t+\mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{r}})\rho_{\mathbf{p},\mathbf{r},t}=-i[\mathsf{H}_{\mathbf{p},\mathbf{r},t},\rho_{\mathbf{p},\mathbf{r},t}],\tag{1}$$

where $\mathsf{H}_{\mathbf{p},\mathbf{r},t}=\sqrt{2}G_{\rm F}\sum_{\mathbf{p}'}\rho_{\mathbf{p}',\mathbf{r},t}(1-\cos\theta_{\mathbf{p},\mathbf{p}'})$ is a matrix driving flavor evolution by neutrino-neutrino refraction. The advection term proportional to $\mathbf{v}=\mathbf{p}/|\mathbf{p}|$ quantifies a drift in coordinate space caused by inhomogeneities. This term allows for $\mathbf{p}$-dependent instabilities, so even an initially homogeneous system can develop flavor disturbances growing exponentially and drifting to ever smaller scales [15,29–34].

As detailed later, even an elementary example of two colliding beams consisting initially of ν_e and ν_x reveals that the refractive interaction energy changes dramatically as soon as initial seeds of inhomogeneity grow. The solution of this puzzle is that one cannot look at flavor evolution in isolation. Inhomogeneities in the weak potential exert a force and trade refractive interaction energy with the huge reservoir of neutrino kinetic energy. We here derive the missing terms in Eq. (1) and show that to lowest order in a gradient expansion, the energy exchange can be exactly accounted for.

Interaction energy.—We first define the ν–ν interaction energy. The starting point is the many-body Hamiltonian $\mathcal{H}=\mathcal{H}_0+\mathcal{U}$, where $\mathcal{H}_0=\sum_{\alpha,\mathbf{p}}\epsilon_{\mathbf{p}}a^{\dagger}_{\alpha,\mathbf{p}}a_{\alpha,\mathbf{p}}$ is the kinetic energy operator with $\epsilon_{\mathbf{p}}=|\mathbf{p}|$ in the massless limit. Moreover, the interaction energy is

$$\mathcal{U}=\frac{\sqrt{2}G_{\rm F}}{8}\sum_{\{\mathbf{p}\},\alpha,\beta}a^{\dagger}_{\alpha,\mathbf{p}_1}a_{\alpha,\mathbf{p}_2}a^{\dagger}_{\beta,\mathbf{p}_3}a_{\beta,\mathbf{p}_4}\bar{u}_{\mathbf{p}_1}\gamma^{\mu}u_{\mathbf{p}_2}\bar{u}_{\mathbf{p}_3}\gamma_{\mu}u_{\mathbf{p}_4},\tag{2}$$

where $G_{\rm F}$ is Fermi's constant, $\sum_{\{\mathbf{p}\}}$ is performed over all momenta such that $\mathbf{p}_1+\mathbf{p}_3=\mathbf{p}_2+\mathbf{p}_4$, and $u_{\mathbf{p}}$ is the spinor of a left-handed massless neutrino with momentum $\mathbf{p}$, normalized such that $\bar{u}_{\mathbf{p}}\gamma^0u_{\mathbf{p}}=2$.

The energy of the system is the expectation value of $\mathcal{H}$ over the quantum state. In particular, the kinetic energy is

$K = \langle \mathcal{H}_0 \rangle = \int d^3\mathbf{r} \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} \mathrm{Tr}(\rho_{\mathbf{p}})$. For the interaction energy, we need to determine the average value of a string of four creation and annihilation operators. We use mean field approximation, where the quantum state is assumed to be the product of single-particle states. The expectation value factorizes as $\langle a_1^\dagger a_2 a_3^\dagger a_4 \rangle = \langle a_1^\dagger a_2 \rangle \langle a_3^\dagger a_4 \rangle + (\delta_{2,3} - \langle a_3^\dagger a_2 \rangle) \langle a_1^\dagger a_4 \rangle$. In this way we define a potential energy of interaction $U = \langle \mathcal{U} \rangle$. For a homogeneous system, it is

$$U = \frac{\sqrt{2} G_{\mathrm{F}}}{2} \int d^3\mathbf{r} \sum_{\mathbf{p},\mathbf{p}'} \left[\mathrm{Tr}(\rho_{\mathbf{p}}) \mathrm{Tr}(\rho_{\mathbf{p}'}) + \mathrm{Tr}(\rho_{\mathbf{p}} \rho_{\mathbf{p}'}) \right] \times (1 - \cos\theta_{\mathbf{p},\mathbf{p}'}), \tag{3}$$

where we neglect an irrelevant (infinite) renormalization of the neutrino chemical potential.

In the two-flavor case, the density matrices are often decomposed into their trace and trace-free part as

$$\rho_{\mathbf{p}} = \frac{P^0_{\mathbf{p}} \mathbb{1} + \vec{P}_{\mathbf{p}} \cdot \vec{\sigma}}{2}, \tag{4}$$

where $\mathbb{1}$ is a unit matrix, $P^0_{\mathbf{p}} = \mathrm{Tr}(\rho_{\mathbf{p}})$, $\vec{\sigma}$ a vector of Pauli matrices, and $\vec{P}_{\mathbf{p}}$ a polarization vector. ($\vec{P}$ is a vector in flavor space, $\mathbf{p}$ one in phase space.) Exactly one particle in mode $\mathbf{p}$ implies $P^0_{\mathbf{p}} = |\vec{P}_{\mathbf{p}}| = 1$. Only $\vec{P}_{\mathbf{p}}$ evolves by oscillations, whereas the trace (total particle number in a $\mathbf{p}$ mode) is conserved. The oscillatory part of the interaction energy is

$$U_{\mathrm{osc}} = \frac{\sqrt{2} G_{\mathrm{F}}}{4} \int d^3\mathbf{r} \sum_{\mathbf{p},\mathbf{p}'} (1 - \cos\theta_{\mathbf{p},\mathbf{p}'}) \vec{P}_{\mathbf{p}} \cdot \vec{P}_{\mathbf{p}'}, \tag{5}$$

corresponding to that part of Eq. (3) that depends on the trace-free parts of the ρ matrices.

For simplicity we assume in the following a system that remains homogeneous in all but the spatial z direction and use $v = v_z$. The standard EOM Eq. (1) falls into pieces for particle number and flavor polarization as

$$\partial_t P^0_{\mathbf{p}} + v \partial_z P^0_{\mathbf{p}} = 0, \tag{6a}$$

$$\partial_t \vec{P}_{\mathbf{p}} + v \partial_z \vec{P}_{\mathbf{p}} = \sqrt{2} G_{\mathrm{F}} (\vec{P}_0 - v \vec{P}_1) \times \vec{P}_{\mathbf{p}}, \tag{6b}$$

where we use the angular moments $\vec{P}_0 = \sum_{\mathbf{p}} \vec{P}_{\mathbf{p}}$ and $\vec{P}_1 = \sum_{\mathbf{p}} v \vec{P}_{\mathbf{p}}$. Thus, the usual equations decouple the neutrino number from their flavor evolution.

Different from a neutrino gas subject to collisions with matter (see, e.g., Refs. [35–44]), where energy is not conserved anyway, Eqs. (6) describe a closed system that admits a conserved energy. The kinetic part $K = \int d^3\mathbf{r} \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} P^0_{\mathbf{p}}$ is seen to be conserved from Eq. (6a). Also the nonoscillatory part of the interaction energy is separately conserved. $U_{\mathrm{osc}} = (\sqrt{2} G_{\mathrm{F}}/4) \int d^3\mathbf{r} (\vec{P}_0^2 - \vec{P}_1^2)$, which follows from Eq. (5), must then be conserved as well. In the homogeneous case, this is indeed the case; the dynamics is periodic and described by a fictitious pendulum whose conserved energy coincides with U_{osc} [45–48]. However, Eq. (6b) immediately reveals that the conservation breaks down for inhomogeneous settings

$$\frac{dU_{\mathrm{osc}}}{dt} = -\frac{\sqrt{2} G_{\mathrm{F}}}{2} \int d^3\mathbf{r} (\vec{P}_0 \cdot \partial_z \vec{P}_1 - \vec{P}_1 \cdot \partial_z \vec{P}_2), \tag{7}$$

where $\vec{P}_2 = \sum_{\mathbf{p}} v^2 \vec{P}_{\mathbf{p}}$. Therefore, the traditional EOMs are not consistent.

Two-beam example.—The nonconservation of U_{osc} is a large effect as we show in a simple example of two opposite beams along the z axis ($v = \pm 1$). The energies and number densities are equal, one beam initially occupied with ν_e, the other with ν_x. As usual, the beams represent many neutrinos with $\mathbf{p}$'s close enough that the small spread is unimportant on the relevant timescales. Therefore, we represent the flavor polarization of each beam by a highly occupied ρ matrix and concomitant polarization vector $\vec{P}_\pm(z,t)$. We thus need to solve

$$(\partial_t + \partial_z) \vec{P}_+ = 2 \vec{P}_- \times \vec{P}_+, \tag{8a}$$

$$(\partial_t - \partial_z) \vec{P}_- = 2 \vec{P}_+ \times \vec{P}_-, \tag{8b}$$

where the interaction strength was absorbed in the units of time and space. For $\partial_z \vec{P}_\pm = 0$ there is no instability, but the system has unstable inhomogeneous modes.

We can show energy nonconservation analytically in the initial linear regime. In this limit, it is convenient to express the x–y part of the polarization vector (the off-diagonal element of the density matrix) in the form $\psi_\pm = P^x_\pm + i P^y_\pm$ and initially we take $P^z_\pm(z,0) = \zeta_\pm$ to be constant. The linearized EOMs ($|\psi| \ll |\zeta|$) are

$$(\partial_t + \partial_z) \psi_+ = 2i(+\zeta_- \psi_+ - \zeta_+ \psi_-), \tag{9a}$$

$$(\partial_t - \partial_z) \psi_- = 2i(-\zeta_- \psi_+ + \zeta_+ \psi_-). \tag{9b}$$

These EOMs are most easily solved for the spatial Fourier modes $\tilde{\psi}_\pm(k,t) = L^{-1} \int_{-L/2}^{+L/2} dz\, \psi_\pm(z,t) e^{-ikz}$ for a periodic box of length L where $k = 2\pi n/L$ with integer n is a discrete wave vector. Equations (9) shows that a k mode is unstable if $k_1 < k < k_2$ with $k_{1,2} = \zeta_- - \zeta_+ \mp 2\sqrt{-\zeta_- \zeta_+}$. The corresponding eigenfrequencies are $\omega_k = \omega_0 \pm i\gamma_k$, where the precession frequency $\omega_0 = -\zeta_- - \zeta_+$ does not depend on k and the growth rate is $\gamma_k = \sqrt{(k - k_1)(k_2 - k)}$. In particular, the maximum growth

rate is attained for $\bar{k} = \zeta_- - \zeta_+ = (k_1 + k_2)/2$ and is $\bar{\gamma} = 2\sqrt{-\zeta_+\zeta_-} = (k_2 - k_1)/2$.

The oscillation energy $U_{\rm osc} = \int dz \vec{P}_+ \cdot \vec{P}_-$ has one piece from the initial z component, $U_{\rm osc}(0) = \zeta_+\zeta_- L$ that is conserved in the linear regime, and a piece from the off-diagonal terms $\Delta U_{\rm osc} = \int dz(\psi_+\psi_-^* + \psi_+^*\psi_-)/2 = (L/2)\sum_k(\tilde{\psi}_{+,k}\tilde{\psi}_{-,k}^* + \tilde{\psi}_{+,k}^*\tilde{\psi}_{-,k})$, where in the continuous limit $\sum_k = \int dk/2\pi$. This piece can grow exponentially if there are small seeds for the unstable Fourier modes. If we decompose the $v = \pm 1$ modes in terms of the eigenmodes of the linear analysis, the k modes evolve as

$$\begin{pmatrix} \tilde{\psi}_{+,k}(t) \\ \tilde{\psi}_{-,k}(t) \end{pmatrix} = \alpha_k \begin{pmatrix} k + \omega_0 + i\gamma_k \\ k - \omega_0 - i\gamma_k \end{pmatrix} e^{(\gamma_k - i\omega_0)t} + \beta_k \begin{pmatrix} k + \omega_0 - i\gamma_k \\ k - \omega_0 + i\gamma_k \end{pmatrix} e^{-(\gamma_k + i\omega_0)t}. \quad (10)$$

Matching this expression with the initial conditions $\tilde{\psi}_{\pm,k}(0) = \tilde{\psi}^0_{\pm,k}$ reveals for the growing piece

$$\alpha_k = \frac{(\gamma_k + i\omega_0)\left(\tilde{\psi}^0_{+,k} + \tilde{\psi}^0_{-,k}\right) - ik\left(\tilde{\psi}^0_{+,k} - \tilde{\psi}^0_{-,k}\right)}{4k\gamma_k}. \quad (11)$$

Substituting in the transverse part of the oscillation energy, we finally find

$$\Delta U_{\rm osc}(t) = L\sum_{k=k_1}^{k_2} 2k(k - \bar{k})|\alpha_k|^2 e^{2\gamma_k t}, \quad (12)$$

where we restrict the summation to the range of unstable eigenmodes. Notice that the most unstable eigenmode $k = \bar{k}$ does not contribute to energy nonconservation due to the vanishing prefactor. Otherwise, an exponential growth appears already at the linear level.

For more than two beams, from Eq. (7) it follows that $dU_{\rm osc}/dt \propto \sum_k kv\vec{P}_v(k) \cdot \vec{P}^*_{v'}(k)(1 - vv')$, and thus it is still true that the energy change grows with $e^{2\bar{\gamma}t}$, provided that $k \neq 0$; homogeneous modes, with $k = 0$, do not lead to energy change due to the prefactor kv.

For a numerical solution that extends to the nonlinear regime, we use periodic boundary conditions on a box of length $L = 100$, divided in N spatial bins. As initial conditions, we use $P^z_\pm(0) = \zeta_\pm = \pm 1/2$. These parameters imply that the range of unstable modes is delimited by $k_1 = -2$ and $k_2 = 0$, the maximum growth rate obtains for $\bar{k} = -1$ and is $\bar{\gamma} = 1$, and the initial oscillation energy is $U_{\rm osc}(0) = \zeta_-\zeta_+ L = -25$. The transverse components $P^{x,y}_\pm$ are seeded with randomly sampled functions with a spatial dependence

$$P^{x,y}_\pm = \sum_{n=-N_{\rm max}}^{N_{\rm max}} c^{x,y}_{\pm,n} e^{i\phi^{x,y}_{\pm,n} + i\frac{2\pi nz}{L}}, \quad (13)$$

where the amplitudes are sampled from a normal distribution with a variance $\sigma^2 = 10^{-8}$ and the phases are uniformly sampled from 0 to 2π; the functions are real and so $c_n = c_{-n}$ and $\phi_n = -\phi_{-n}$. We use arbitrarily $N_{\rm max} = 100$ to avoid seeds at too small scales. Numerical stability requires keeping fluctuations at the scale of the grid spacing as small as possible.

Figure 1 shows the solution for a single realization of initial seeds. The upper panel shows contours of the ν_x content of the beam initially occupied with ν_e, as a function of z and t. The lower panel shows $U_{\rm osc}(t)$ in units of $U_{\rm osc}(0)$. On the shown timescale, the range of unstable modes $-2 < k < 0$ grows nonlinear and then keeps oscillating, with beats between different modes causing an irregular pattern that in detail depends on the choice of seeds. In the longer run, these nonlinear modes feed higher-k modes and flavor variations will obtain on ever smaller scales. If we had used a larger box, or equivalently averaged over more than one realization of the initial conditions, the amplitude of the oscillations in the final state would shrink. Nevertheless, the bulk of the nonconservation of $U_{\rm osc}$ happens in the initial phase, and is clearly a large effect.

Inhomogeneous kinetic equations.—The lack of refractive-energy conservation questions the validity of the traditional EOMs. The strategy for their derivation is a perturbative expansion in three small parameters: (i) the mass-to-energy ratio $r_m = m_\nu/\epsilon_{\mathbf{p}}$, where m_ν is the neutrino

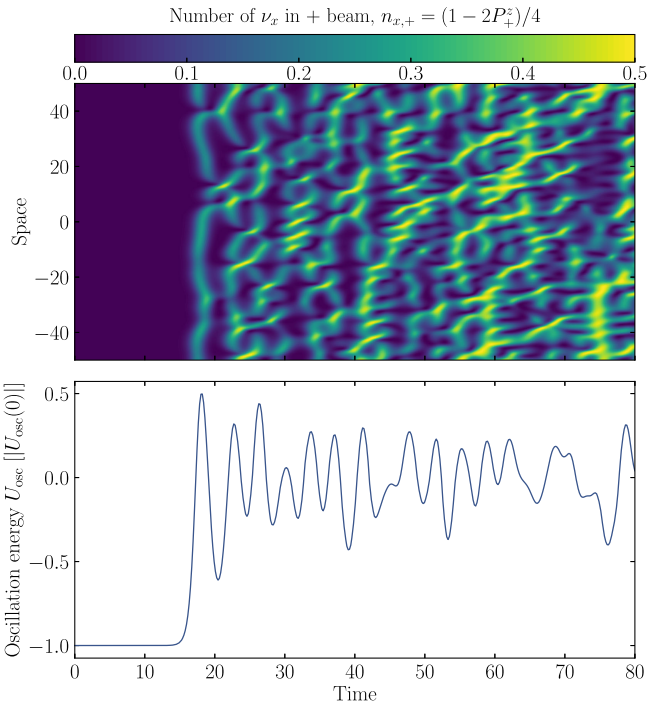

FIG. 1. Solution of our two-beam example for one realization of initial seeds. Top: contours of ν_x content of $v = +1$ beam as a function of z and t. Bottom: evolution of oscillation energy $U_{\rm osc}(t)$ in units of $U_{\rm osc}(0)$.

mass, but in our fast flavor limit we do not worry about it. (ii) The fractional refractive energy shift $r_\mu = \mu/\epsilon_{\mathbf{p}}$, where $\mu = \sqrt{2}G_{\rm F}n_\nu$ is a scale for the ν–ν interaction energy. For $T \simeq 5$ MeV in the decoupling region of a supernova, $n_\nu \simeq 10^{33}$ cm^{-3}, $\mu \simeq 0.2$ meV, and $r_\mu \simeq 10^{-11}$ and thus indeed very small. (iii) The scale of density variations ℓ in units of the neutrino de Broglie wavelength $r_\ell = (\ell|\mathbf{p}|)^{-1} \ll 1$. Density variations are on scales much larger than $|\mathbf{p}|^{-1}$. The EOMs are typically derived to lowest order in all of these small ratios, notably neglecting terms of the order of $r_\ell r_\mu$, which causes the nonconservation of energy.

To augment Eq. (1) with the missing terms, we note that in an inhomogeneous setting, the space-dependent occupation-number matrix is defined by a Wigner distribution [49], the expectation value of

$$\mathcal{D}_{\alpha\beta}(\mathbf{r},\mathbf{p}) = \sum_{\mathbf{k}} a^\dagger_{\beta,\mathbf{p}+\mathbf{k}/2} a_{\alpha,\mathbf{p}-\mathbf{k}/2} e^{-i\mathbf{k}\cdot\mathbf{r}}. \tag{14}$$

Here, the Fourier wave vector $\mathbf{k}$ measures the typical length scale over which the density matrix is changing, and is therefore $|\mathbf{k}| \sim \ell^{-1} \ll |\mathbf{p}|$. The time derivative is obtained using Heisenberg's equations. The commutator with the kinetic energy is easily established as

$$\begin{aligned}\left(\frac{\partial \mathcal{D}_{\alpha\beta}}{\partial t}\right)_0 &= i\sum_{\mathbf{k}} a^\dagger_{\beta,\mathbf{p}+\frac{\mathbf{k}}{2}} a_{\alpha,\mathbf{p}-\frac{\mathbf{k}}{2}} e^{-i\mathbf{k}\cdot\mathbf{r}} \left(\epsilon_{\mathbf{p}+\frac{\mathbf{k}}{2}} - \epsilon_{\mathbf{p}-\frac{\mathbf{k}}{2}}\right) \\ &\simeq -\mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{r}} \mathcal{D}_{\alpha\beta},\end{aligned} \tag{15}$$

recovering the usual advection term.

For the interaction part, we here outline the general strategy and report details in the Supplemental Material [50]. The commutator $[\mathcal{D},\mathcal{U}]$ produces strings of four creation and annihilation operators, or four-point correlation functions. The average of these combinations are evaluated again using the mean field approximation. In the homogeneous limit, when taking the expectation value of $\langle a_1^\dagger a_2\rangle$, the momenta of state 1 and 2 must be equal. We go one step further and include the inhomogeneity to first order in $\mathbf{k}$. The final result, fully derived in the SM, is

$$\begin{aligned}&\partial_t \rho_{\mathbf{p},\mathbf{r}} + \mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{r}}\rho_{\mathbf{p},\mathbf{r}} + \frac{1}{2}\left\{\boldsymbol{\nabla}_{\mathbf{p}}\Omega^{(0)}_{\mathbf{p},\mathbf{r}}, \boldsymbol{\nabla}_{\mathbf{r}}\rho_{\mathbf{p},\mathbf{r}}\right\} \\ &\quad - \frac{1}{2}\left\{\boldsymbol{\nabla}_{\mathbf{r}}\Omega^{(0)}_{\mathbf{p},\mathbf{r}}, \boldsymbol{\nabla}_{\mathbf{p}}\rho_{\mathbf{p},\mathbf{r}}\right\} = i\left[\rho_{\mathbf{p},\mathbf{r}}, \Omega_{\mathbf{p},\mathbf{r}}\right],\end{aligned} \tag{16}$$

where we introduce

$$\Omega^{(0)}_{\mathbf{p},\mathbf{r}} = \sqrt{2}G_{\rm F}\sum_{\mathbf{p}'}(1-\cos\theta_{\mathbf{p},\mathbf{p}'})\left[\rho_{\mathbf{p}',\mathbf{r}} + {\rm Tr}(\rho_{\mathbf{p}',\mathbf{r}})\mathbb{1}\right] \tag{17}$$

that coincides with the standard $\mathsf{H}_{\mathbf{p},\mathbf{r}}$ stated after Eq. (1) except for the additional trace term that drops out in commutators, but not in the anticommutators on the left-hand side. Moreover,

$$\begin{aligned}\Omega_{\mathbf{p},\mathbf{r}} = \Omega^{(0)}_{\mathbf{p},\mathbf{r}} + \sqrt{2}G_{\rm F}\sum_{\mathbf{p}'}\frac{p+p'}{p^2p'^2}(\mathbf{p}\times\mathbf{p}') \\ \cdot\left[\boldsymbol{\nabla}_{\mathbf{r}}\rho_{\mathbf{p}',\mathbf{r}} + {\rm Tr}(\boldsymbol{\nabla}_{\mathbf{r}}\rho_{\mathbf{p}',\mathbf{r}})\mathbb{1}\right],\end{aligned} \tag{18}$$

where $p = |\mathbf{p}|$ and $p' = |\mathbf{p}'|$.

$\Omega_{\mathbf{p},\mathbf{r}}$ is the renormalized quasiparticle energy, in general a matrix in flavor space, analogous to the renormalized quasiparticle energy in Landau's theory of a Fermi liquid [51]. This induces a renormalization in the group velocity $\boldsymbol{\nabla}_{\mathbf{p}}\Omega^{(0)}_{\mathbf{p},\mathbf{r}}$ (only the zero order terms must be kept which already contain one gradient) and a weak force field $-\boldsymbol{\nabla}_{\mathbf{r}}\Omega^{(0)}_{\mathbf{p},\mathbf{r}}$.

All additional terms are small, of the order of $r_\mu r_l$, and thus will not produce *quantitatively* large flavor-conversion effects. However, as a *qualitatively* new effect, the renormalized group velocity causes a slow spatial drift of neutrinos; the order of magnitude of the velocity is $|\boldsymbol{\nabla}_{\mathbf{p}}\Omega^{(0)}_{\mathbf{p},\mathbf{r}}| \sim r_\mu$. Since $r_\mu \sim 10^{-11}$, this is completely negligible compared to the standard neutrino velocity.

The most significant impact comes from the weak force $-\boldsymbol{\nabla}_{\mathbf{r}}\Omega^{(0)}_{\mathbf{p},\mathbf{r}}$, which can change the neutrino kinetic energy. Part of this force originates from gradients in the neutrino number density. Our new insight is that an additional component originates from gradients in flavor composition. To see this clearly, we rewrite the EOMs as

$$\partial_t\rho_{\mathbf{p},\mathbf{r}} + \mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{r}}\rho_{\mathbf{p},\mathbf{r}} = i[\rho_{\mathbf{p},\mathbf{r}},\Omega_{\mathbf{p},\mathbf{r}}] + \boldsymbol{\nabla}_{\mathbf{r}}\cdot\boldsymbol{\Phi}_{\mathbf{p},\mathbf{r}} - \boldsymbol{\nabla}_{\mathbf{p}}\cdot\mathbf{F}_{\mathbf{p},\mathbf{r}}, \tag{19}$$

where

$$\begin{aligned}\boldsymbol{\Phi}_{\mathbf{p},\mathbf{r}} = \frac{\sqrt{2}G_{\rm F}}{2}\sum_{\mathbf{p}'}(1-\cos\theta_{\mathbf{p},\mathbf{p}'}) \\ \times\left[\{\rho_{\mathbf{p}',\mathbf{r}}, \boldsymbol{\nabla}_{\mathbf{p}}\rho_{\mathbf{p},\mathbf{r}}\} + 2{\rm Tr}(\rho_{\mathbf{p}',\mathbf{r}})\boldsymbol{\nabla}_{\mathbf{p}}\rho_{\mathbf{p},\mathbf{r}}\right]\end{aligned} \tag{20}$$

is the number flux of neutrinos passing through a surface element of coordinate space at phase-space location $\{\mathbf{p},\mathbf{r}\}$, as seen from the structure analogous to a continuity equation, whereas $\mathbf{F}_{\mathbf{p},\mathbf{r}}$ is the same in momentum space, given by the same expression with $\boldsymbol{\nabla}_{\mathbf{p}} \to \boldsymbol{\nabla}_{\mathbf{r}}$. This term couples the trace of the density matrix with the polarization vector; taking the trace, we find

$$\begin{aligned}{\rm Tr}(\mathbf{F}_{\mathbf{p},\mathbf{r}}) = \frac{\sqrt{2}G_{\rm F}}{2}\sum_{\mathbf{p}'}(1-\cos\theta_{\mathbf{p},\mathbf{p}'}) \\ \times\left[3P^0_{\mathbf{p}',\mathbf{r}}\boldsymbol{\nabla}_{\mathbf{r}}P^0_{\mathbf{p},\mathbf{r}} + \vec{P}_{\mathbf{p}',\mathbf{r}}\cdot\boldsymbol{\nabla}_{\mathbf{r}}\vec{P}_{\mathbf{p},\mathbf{r}}\right].\end{aligned} \tag{21}$$

Thus, spatial gradients of the polarization vectors can feed ν kinetic energy. This term implies a net rate of energy gain or loss of order $d\epsilon_{\mathbf{p}}/dt \simeq \mu/\ell$. Over a timescale ℓ, the energy that a ν can accumulate is of order $\mu \ll \epsilon_{\mathbf{p}}$, so we recover that this is a small effect *relative* to $\epsilon_{\mathbf{p}}$, but large relative to the interaction energy. It provides the missing channel by which kinetic and refractive energy can be traded. Even a small amount of energy μ lost (gained) from the large ν kinetic energy explains the large change in the refractive energy, since $U_{\rm osc}/K \simeq \mu/\epsilon_{\mathbf{p}}$.

With the new kinetic equations, we can directly prove that to first order in r_μ and r_ℓ, the total energy is conserved. After integrating by parts, we find

$$\frac{dK}{dt} = \int d^3\mathbf{r} \sum_{\mathbf{p}} \mathrm{Tr}(\mathbf{v} \cdot \mathbf{F}_{\mathbf{p},\mathbf{r}}). \tag{22}$$

Substituting the polarization vector part of Eq. (21) in Eq. (22), we find that it precisely balances the rate of change in the oscillation energy found in Eq. (7).

Discussion.—We have shown that the usual kinetic equation for mixed neutrinos Eq. (1) is nonconservative in the inhomogeneous case. We have illustrated this point with a simple two-beam example, where the ν–ν interaction energy strongly changes, and we have also shown this effect analytically in the linear regime. Deriving missing gradient terms beyond Eq. (1), we have shown that the refractive energy gained or lost is precisely traded with neutrino kinetic energy that usually is not followed. In our two-beam example, the monochromatic initial energy distribution develops small space-time dependent shifts that account for the missing energy. We have not tried to study this effect numerically because it requires many $\mathbf{p}$ modes around the original one, but we have proven energy conservation analytically.

The usual EOMs correctly account for flavor evolution, meaning the trace-free part of the density matrices often described by polarization vectors. On the other hand, the particle number in a given $\mathbf{p}$ mode is conserved, corresponding to the trace part of the EOM, where the left-hand side of Eq. (1) is a continuity equation. Therefore, the trace part of the EOM must be expanded to higher order in the gradients to capture nontrivial evolution. The reshuffling of neutrinos among $\mathbf{p}$ modes is a small effect relative to the large kinetic energies, yet precisely absorbs the missing refractive energy.

These novel terms come from the gradients of the neutrino self-energy. With hindsight, that they would affect neutrino evolution, is obvious from the viewpoint of physical kinetics, and implicitly present in previous formal derivations [2,3,5–7], but usually assumed to be a small effect. Our new insight is that fast conversions spontaneously break homogeneity, magnifying the gradients of the trace-free density matrix, making these terms large enough to explain completely the apparent nonconservation of energy.

We note that Eq. (1) conserves entropy and including the new gradient terms, this is also the case, i.e., entropy is conserved order by order in the gradient expansion (see Supplemental Material). On the practical level, the conservation of entropy, but not of energy, could be used to test numerical stability. Even more importantly, in making predictions on the final state induced by conversions, energy conservation cannot be used in practice, unless the new terms are kept.

Our finding may shed new light on a recent proposal that the final outcome of fast conversions may be some sort of thermalized state [52]. The quasisteady state generically observed in numerical simulations of fast conversions [53–58] is only determined by the oscillatory part of the density matrix, yet its dynamics does not admit a separately conserved energy. Thus, it cannot separately thermalize, since it exchanges energy with the kinetic energy of neutrinos.

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