Energetic Cost for Speedy Synchronization in Non-Hermitian Quantum Dynamics

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Quantum synchronization is crucial for understanding complex dynamics and holds potential applications in quantum computing and communication. Therefore, assessing the thermodynamic resources required for finite-time synchronization in continuous-variable systems is a critical challenge. In the present work, we find these resources to be extensive for large systems. We also bound the speed of quantum and classical synchronization in coupled damped oscillators with non-Hermitian anti-$\mathcal{PT}$-symmetric interactions, and show that the speed of synchronization is limited by the interaction strength relative to the damping. Compared to the classical limit, we find that quantum synchronization is slowed by the noncommutativity of the Hermitian and anti-Hermitian terms. Our general results could be tested experimentally, and we suggest an implementation in photonic systems.

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The study of synchronization originated in the 17th century, when Huygens noted the gradual buildup of correlations in coupled pendula [1]. Similar behavior has been found ubiquitously, such as in many-body physics, biology, and even human activities [2–10]. As the coordination of multiple objects implies some kind of communication, synchronization is a key mechanism for establishing order in complex systems [11–16]. Thus, it is also of interest in thermodynamics and quantum information theory [17–23], where it has become an emerging research focus due to potential applications in quantum computing and communication [24,25].

In quantum dynamics, the primary focus has been on synchronization in discrete systems [26–32], whereas continuous-variable models are often treated classically [33–35]. However, to study the quantum limit of classical models, genuine continuous-variable scenarios are required [5,36–40]. Multiple ways of quantifying synchronization have been devised for both discrete and continuous-variable quantum systems [37,39,41–45]; however there is no clear consensus as to which metric is universally applicable. Moreover, existing work also provides only limited insight into the timescales and energy scales on which the process occurs.

Even though some studies have taken into account the complete quantum (transient and steady state) dynamics to better understand synchronization [20,38,46–49], these works have not focused on the resources required. On the other hand, quantum speed limits [50–60] seem to be the ideal to quantify the (dynamical) resources required, and they have recently been used to constrain the rate of synchronization [61,62].

In this Letter, we apply quantum speed limits and quantum thermodynamics to a general model of continuous-variable quantum dynamics. A measure of complete synchronization is defined, which is scale invariant and is sufficient for phase synchronization. We obtain relations of the degree of synchronization with the distance from thermodynamic equilibrium, resulting in an extensive expression for the minimal work necessary for synchronization.

Specifically, we study a quantum master equation that includes both non-Hermitian dynamics and a dissipative term in the Gorini-Kassakowski-Sudarshan-Lindblad (GKSL) form, resulting in a nonlinear dynamical semigroup. We find that the rate of synchronization is determined by a competition between the irreversible entropy production caused by damping, which slows synchronization, and the strength of the anti-Hermitian coupling, which speeds up synchronization. The resulting upper bound on the synchronization rate has terms of the form of the Mandelstam-Tamm inequality [63], where speed scales with the uncertainty of the energy, except in this case even the uncertainties of the Hermitian and anti-Hermitian parts of the Hamiltonian are crucial. As an example, we consider a dissipatively coupled photonic dimer and find that the quantum system synchronizes in a parameter regime wherein it is impossible for the classical model to synchronize, thereby displaying a quantum advantage.

Measure of synchronization.—We consider N quantum oscillators with annihilation operators $\hat{a}_1, \ldots, \hat{a}_N$. The corresponding dimensionless quadrature operators $\hat{\mathbf{r}} = (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_N, \hat{p}_N)^T$ read as [64–66]

$$\hat{x}_j = \frac{\hat{a}_j + \hat{a}_j^\dagger}{\sqrt{2}}, \qquad \hat{p}_j = \frac{\hat{a}_j - \hat{a}_j^\dagger}{i\sqrt{2}}. \tag{1}$$

For the system to synchronize, the phase space coordinates of the different oscillators need to converge. There can be two distinct types of synchronization: (i) complete synchronization, where the phase space trajectories of multiple subsystems converge, and (ii) phase synchronization, for which the phase angles of multiple subsystems converge [16]. An intuitive measure to characterize complete synchronization of a quantum bipartite system is $\mathcal{S}_c = 2\langle(\hat{\mathbf{r}}_2 - \hat{\mathbf{r}}_1)^2\rangle^{-1}$ [37]. In some special case scenarios, such as amplitude death, the growth of $\mathcal{S}_c$ does not imply synchronization [38]. Therefore, we define a new measure as the distance between the oscillators *relative* to the total radius in phase space. For a bipartite system, we have

$$D^2 \equiv \frac{\langle(\hat{\mathbf{r}}_2 - \hat{\mathbf{r}}_1)^2\rangle}{\langle\hat{\mathbf{r}}^2\rangle}, \tag{2}$$

where, in our notation, $\langle\hat{\mathbf{r}}^2\rangle = \langle\hat{\mathbf{r}}_1^2\rangle + \langle\hat{\mathbf{r}}_2^2\rangle$. Note that the so-defined D is scale invariant with respect to $\hat{\mathbf{r}}$ [67]. The bipartite distance measure can be expressed in terms of angular and radial measures of similarity,

$$D^2 = 1 - \mathcal{S}_r\mathcal{S}_\theta, \tag{3}$$

where

$$\mathcal{S}_\theta = \frac{\langle\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2\rangle}{|\langle\hat{\mathbf{r}}_1\rangle||\langle\hat{\mathbf{r}}_2\rangle|} \equiv \cos\theta, \tag{4}$$

with $|\langle\hat{\mathbf{r}}_j\rangle| = \sqrt{\langle\hat{\mathbf{r}}_j^2\rangle}$ and

$$\mathcal{S}_r = 2\sqrt{\frac{\langle\hat{\mathbf{r}}_1^2\rangle}{\langle\hat{\mathbf{r}}^2\rangle}\left(1 - \frac{\langle\hat{\mathbf{r}}_1^2\rangle}{\langle\hat{\mathbf{r}}^2\rangle}\right)}. \tag{5}$$

The quantity $\mathcal{S}_r$ is similar to the binary entropy function [68], and is maximized when $\langle\hat{\mathbf{r}}_1^2\rangle = \langle\hat{\mathbf{r}}_2^2\rangle$. Equations (4) and (5) reveal that D^2 is between 0 and 2, with values less than 1 indicating synchronization and values greater than 1 indicating antisynchronization. It is also clear from Eq. (3) that for D^2 to become small, both $\mathcal{S}_r$ and $\mathcal{S}_\theta$ must approach their maximal values of 1, implying that we require amplitude and phase synchronization by requiring a decay of D^2 in time. For a system of N oscillators, Eq. (2) can be generalized as

$$D^2 \equiv 2\left(1 - \frac{\langle\bar{\mathbf{r}}^2\rangle}{\langle\overline{\mathbf{r}^2}\rangle}\right), \tag{6}$$

where $\bar{\mathbf{r}} = (\sum_{j=1}^N \hat{x}_j, \sum_{j=1}^N \hat{p}_j)^T/N$ and $\langle\overline{\mathbf{r}^2}\rangle = \langle\mathbf{r}^2\rangle/N^2$. Here, D^2 is non-negative, which follows from Jensen's inequality.

Throughout this work, we consider scenarios in which the N oscillators are initially uncoupled and in contact with a thermal bath at inverse temperature β. Initially, the oscillators are allowed to come to their respective equilibrium states $\hat{\rho}_j^{\mathrm{eq}}$ at inverse temperature β_0, and then a coupling between them is turned on. Hence, $\hat{\rho}_0 = \otimes_{j=1}^N \hat{\rho}_j^{\mathrm{eq}}$ is our initial state. To quantify if our system has synchronized we will require that the distance D becomes small and then *stays* small. In other words, given a D_s, the system synchronizes to within the distance D_s if there exists a time τ such that for all $t \geq \tau$, $D \leq D_s$. The smallest τ for which this holds will be called the synchronization time τ_s.

Dynamics.—We now consider the N oscillators with natural frequencies $\omega_1, \ldots, \omega_N$ and an arbitrary anti-Hermitian coupling whose Hamiltonian reads as

$$\hat{H} = \hat{H}_0 + i\hat{H}_c = \sum_{j=1}^N \omega_j\left(\hat{n}_j + \frac{1}{2}\right) + i\hat{H}_c, \tag{7}$$

where $\hat{H}_c$ is Hermitian coupling Hamiltonian, $\hat{n}_j$ is the number operator $\hat{a}_j^\dagger\hat{a}_j$, and we set $\hbar = 1$. Such non-Hermitian Hamiltonians (7) are effective descriptions of *controlled* dissipation in open quantum systems [69–76]. In addition, our system interacts with a thermal environment [77,78], yielding the following quantum master equation (QME),

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}_0, \hat{\rho}] + \{\hat{H}_c, \hat{\rho}\} - 2\langle\hat{H}_c\rangle\hat{\rho} + \mathcal{D}[\hat{\rho}], \tag{8}$$

where $\langle\hat{O}\rangle = \mathrm{tr}\{\hat{O}\,\hat{\rho}\}$, and the term $-2\langle\hat{H}_c\rangle\hat{\rho}$ is included to preserve normalization [79]. This *nonlinear* equation [80] satisfies the convex quasilinearity condition and the semigroup property making it a valid quantum evolution [81]. The dissipator $\mathcal{D}$ takes the GKSL form $\mathcal{D}[\hat{\rho}] = \sum_i \hat{F}_i\hat{\rho}\hat{F}_i^\dagger - \{\hat{F}_i^\dagger\hat{F}_i, \hat{\rho}\}/2$ [82]. We split the Liouvillian into noninteracting $\mathcal{L}_0[\hat{\rho}]$ and interacting $\mathcal{L}_c[\hat{\rho}]$ parts given by

$$\begin{aligned}\mathcal{L}_0[\hat{\rho}] &= -i[\hat{H}_0, \hat{\rho}] + \mathcal{D}[\hat{\rho}]\\ \mathcal{L}_c[\hat{\rho}] &= \{\hat{H}_c, \hat{\rho}\} - 2\langle\hat{H}_c\rangle\hat{\rho},\end{aligned} \tag{9}$$

such that $d\hat{\rho}/dt = \mathcal{L}(\hat{\rho}) = \mathcal{L}_0(\hat{\rho}) + \mathcal{L}_c(\hat{\rho})$. The state $\hat{\rho}_0$ is a stationary state of the Hermitian $\hat{H}_0$ and Lindblad terms, i.e., $\mathcal{L}_0(\hat{\rho}_0) = 0$. Note that Eq. (8) is nonlinear in $\hat{\rho}$ because of the term $-2\langle\hat{H}_c\rangle\hat{\rho}$, as $\langle\hat{H}_c\rangle$ itself depends linearly on $\hat{\rho}$. We will first examine the case of general $\hat{H}_c$, and later specialize to a specific dimer model which results in a *quantum* Stuart-Landau equation.

Quantum synchronization far from equilibrium.—As the system evolves, it will depart from the initial state $\hat{\rho}_0$ due to the anti-Hermitian coupling. Let $\hat{\rho}_{G,E}$ denote a Gibbs state of the uncoupled system with energy E, $\hat{\rho}_{G,E} \propto \exp(-\beta_E\hat{H}_0)$, and $\mathrm{tr}\{\hat{\rho}_{G,E}\hat{H}_0\} = E$. We introduce a measure χ, an *ergotropy* [83,84] *of synchronization*, to quantify the departure of the reduced state $\hat{\rho}$ from the set of Gibbs states of the uncoupled system,

$$\chi \equiv \min_U S(\hat{\rho} \| \hat{\rho}_{G,U}), \tag{10}$$

in terms of the quantum relative entropy $S(\hat{\rho}_1 \| \hat{\rho}_2) = \mathrm{tr}\{\hat{\rho}_1(\ln \hat{\rho}_1 - \ln \hat{\rho}_2)\}$ [85]. The minimization over U in Eq. (10) means that χ is a property of the state $\hat{\rho}$ and does not depend on the bath parameters, which is crucial for establishing a meaningful relationship between the (bath independent) degree of synchronization and χ. In the Supplemental Material [86], we show that for the distance measure D to become small, χ must become large, and we generally have

$$\chi \geq -2(N-1)\ln\left(\frac{1}{2}e^{1/(N-1)}\sqrt{\frac{(N/\kappa)^{N/(N-1)}}{2(N-1)}}D\right), \tag{11}$$

where $\kappa = \omega_{\min}/\omega_{\max}$, and for $N = 2$ the above general expression reduces to

$$\chi \geq -2\ln\left(\frac{eD}{\sqrt{2}\kappa}\right). \tag{12}$$

The analogous bound for classical bipartite systems (see the Supplemental Material [86]) reads as [92]

$$\chi^{(\mathrm{cl})} \geq -2\ln\left(\frac{\sqrt{2}D}{\kappa}\right), \tag{13}$$

which is defined in terms of the classical relative entropy [94,95]. In what follows we refer to the right-hand side of Eq. (11) as $\chi_{\min}$ [96].

A sample of random two-mode Gaussian states is shown in Fig. 1 and compared to Eqs. (12) and (13). Evidently there is a region between the bounds where there may exist states exhibiting a *quantum advantage*, although such states are not present in our random sample. For example, we see that for $D = 0.5$, the classical bound requires χ to be almost unity, whereas the more permissive quantum bound requires χ to be only slightly greater than zero. If such states exist with χ much less than unity for $D = 0.5$, the quantum analysis yields a lower cost [see Eq. (18)] to achieve the same degree of synchronization.

We quantify the departure from the initial equilibrium state $\hat{\rho}_0$ by introducing a parameter,

$$L = S(\hat{\rho} \| \hat{\rho}_0). \tag{14}$$

Unlike χ that quantifies the distance of the reduced state $\hat{\rho}$ from all possible Gibbs states and *minimizes* over the energy, the parameter L measures the distance with respect to only one specific Gibbs state given by the initial condition. Given the distance measures χ and L, the following chain of inequalities follows:

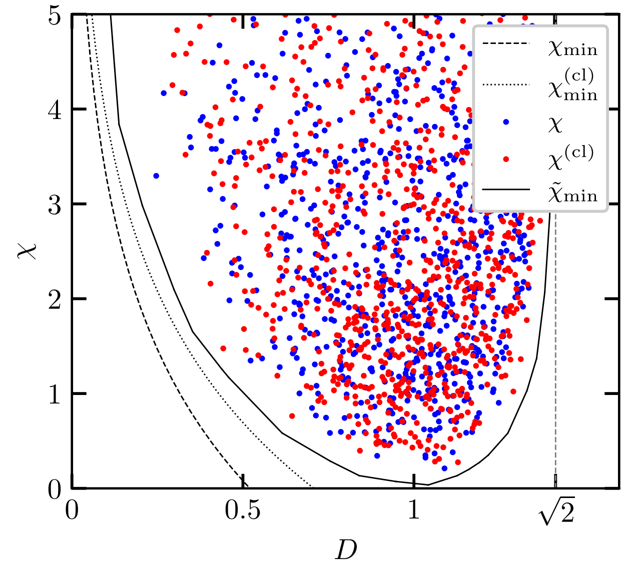

FIG. 1. Quantum ($\chi_{\min}$) and classical ($\chi_{\min}^{(\mathrm{cl})}$) lower bounds on χ, and convex hull ($\tilde{\chi}_{\min}$) of 10^6 random Gaussian states (1000 states plotted as colored circles). Convex hull is the same for quantum and classical sample states.

$$L \geq \chi \geq \Lambda. \tag{15}$$

Here $\Lambda = \min_{\hat{\sigma}\in\Omega} S(\hat{\rho}\|\hat{\sigma})$ is the relative entropy of entanglement with Ω being the set of all separable states of the system [97]. By assumption, $L = \chi = 0$ at time $t = 0$. Therefore as a consequence of Eq. (11), for the system to synchronize in time τ_s to distance D_s we must have

$$L \geq \chi_{\min}(N, \kappa, D_s). \tag{16}$$

Also note that [86] $\dot{L} = \beta\dot{E} - \dot{S}$, so if we write the first law of thermodynamics as $\dot{E} = \dot{W} - \dot{Q}$, and the second law as $\dot{S} + \beta\dot{Q} \geq 0$ [98–102], we have

$$\dot{W} \geq \frac{1}{\beta}\dot{L}. \tag{17}$$

Here $S = -\mathrm{tr}\{\hat{\rho}\ln\hat{\rho}\}$ is the von Neumann entropy [103]. Moreover, since $W(0) = L(0) = 0$ it follows that $W \geq L/\beta$, so we have a lower bound on the amount of work required for synchronization, which is our first main result:

$$W \geq \frac{1}{\beta}\chi_{\min}(N, \kappa, D), \tag{18}$$

where $\chi_{\min}(N, \kappa, D)$ is the right-hand side of Eq. (11). Interestingly, the quantity $\chi_{\min}$ is asymptotically linear in N, indicating that the work requirement for synchronization is extensive. The minimum asymptotic work cost

$$W_{\min}^{\infty} = \chi_{\min}^{\infty} = -\ln\left(\frac{D^2}{8\kappa}\right)N, \tag{19}$$

such that

$$\lim_{N\to\infty} \frac{\chi_{\min}}{\chi_{\min}^{\infty}} = 1. \tag{20}$$

Classically, such an asymptotic cost

$$W_{\min}^{(\mathrm{cl})\infty} = \chi_{\min}^{(\mathrm{cl})\infty} = -\ln\left(\frac{D^2}{2\kappa}\right)N, \tag{21}$$

is lower than the quantum case indicating that in the limit of many oscillators, the thermodynamic costs of synchronizing classical systems will always be lower than synchronizing equivalent quantum systems. However for small values of N the asymptotic expressions are invalid, and the classical synchronization cost may be more [104], and we leave the full investigation of such cases to future work.

Rate of quantum synchronization.—The rate of evolution of L can be found directly from Eq. (8) [86],

$$\dot{L} = 2\mathrm{tr}\{(\hat{H}_c - \langle\hat{H}_c\rangle)\hat{\rho}\ln\hat{\rho}\} + 2\beta_0\mathcal{C}_{CE} - \sigma_0, \tag{22}$$

where $\mathcal{C}_{CE} = \frac{1}{2}\langle\hat{H}_c\hat{H}_0\rangle - \langle\hat{H}_c\rangle\langle\hat{H}_0\rangle$ is the covariance of $\hat{H}_0$ and $\hat{H}_c$, and $\sigma_0 = -\mathrm{tr}\{\mathcal{D}[\hat{\rho}](\ln\hat{\rho} - \ln\hat{\rho}_0)\}$ is the nonnegative irreversible entropy production [70,105,106] of the uncoupled system. In the case that $\hat{H}_c$ is an unbounded operater, $\dot{L}$ can be bounded from above [107],

$$\dot{L} \leq 2\Delta_C\sqrt{\mathcal{E} + S_{G,E}^2} + 2\beta_0\sqrt{\Delta_E^2\Delta_C^2 - \frac{1}{2}\left|\langle[\hat{H}_0, \hat{H}_c]\rangle\right|^2} - \sigma_0. \tag{23}$$

Here, we use the von Neumann entropy of the Gibbs state, $S_{G,E} = -\mathrm{tr}\{\hat{\rho}_{G,E}\ln\hat{\rho}_{G,E}\}$, as well as the capacity of entanglement $\mathcal{E} = \mathrm{tr}\{\hat{\rho}(\ln\hat{\rho})^2\} - S^2$, which is the second moment of surprisal [108]. Here, $\Delta_E^2 = \mathrm{tr}\{\hat{\rho}\hat{H}^2\} - \mathrm{tr}\{\hat{\rho}\,\hat{H}\}^2$ and $\Delta_C^2 = \mathrm{tr}\{\hat{\rho}\hat{H}_c^2\} - \mathrm{tr}\{\hat{\rho}\hat{H}_c\}^2$ are the variances of the operators $\hat{H}$ and $\hat{H}_c$ respectively.

A classical limit of the master equation (8) can be derived by identifying a quantum phase space distribution with a classical probability density [109–111], and a corresponding classical bound (see Supplemental Material [86]) reads as

$$\dot{L}^{(\mathrm{cl})} \leq \beta_0\langle\nabla H_c \cdot \nabla H_0\rangle - \langle\nabla^2 H_c\rangle + 2\Delta_C\sqrt{\langle\ln(f)^2\rangle} + 2\beta_0\Delta_C\Delta_E - \sigma_0. \tag{24}$$

Equations (24) and (23) differ because of the presence of the geometric terms associated with phase space flow [112], as well as the absence of the commutator between $\hat{H}_0$ and $\hat{H}_c$. The interpretation of Eq. (23) is as follows: the last term σ_0 is the rate of irreversible entropy production [106] which one would obtain in the absence of coupling, and this term will always be negative. Therefore the other terms must have a net positive effect larger than this entropy production rate in order for synchronization to occur. The first term is of the form of a quantum speed limit with the additional factor of entropy. This term unsurprisingly implies that stronger non-Hermitian coupling leads to faster synchronization. The second term is reminiscent of the Mandelstam-Tamm quantum speed limit [63], and involves the second moments of both the Hermitian and anti-Hermitian parts of the Hamiltonian; however there is a penalty that scales with the square of their commutator. This term arises from the uncertainty relation [113], and can be explained by the fact that synchronization is most effective when there is a large correlation between the observables corresponding to the Hermitian and anti-Hermitian parts of the Hamiltonian.

Coupled waveguide model.—Yang *et al.* [114] proposed an experimentally realizable effective anti-$\mathcal{PT}$ symmetric Hamiltonian with the non-Hermitian coupling of the form

$$\hat{H}_c = \frac{k}{2}\left(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1\right). \tag{25}$$

The two modes are also in contact with heat reservoirs [115], resulting in a master equation (see the Supplemental Material [86]) of the form of Eq. (8) with four jump operators, $\hat{F}_{i-} = \sqrt{\gamma_{i-}}\hat{a}_i^2$, $\hat{F}_{i+} = \sqrt{\gamma_{i+}}(\hat{a}_i^\dagger)^2$ with $\gamma_{i-} = \gamma_i[\bar{n}(2\omega_i) + 1]$ and $\gamma_{i+} = \gamma_i\bar{n}(2\omega_i)$, obeying local detailed balance [116], and $\bar{n}(\omega) = (\exp[\beta\omega] - 1)^{-1}$ being the Bose-Einstein distribution.

In Fig. 2(a) we see that the evolution in the χ-D plane for the dimer model respects the analytically derived bound [Eq. (12)] and is confined within the convex hull of Gaussian states [117]. This also guarantees that the lower bound on work [Eq. (18)] is obeyed. In Fig. 2(b) we see the time evolution of D and χ, and it is notable that χ is almost always monotonically increasing whereas D has more significant reversals in direction. Moreover, as the non-Hermitian coupling strength k increases the system synchronizes faster. This property is also reflected in our bound on the speed of synchronization, Eq. (23). The bound provides a deeper insight into this behavior as a *competition* between the non-Hermitian coupling [first two positive terms on the rhs of Eq. (23) that are proportional to $\hat{H}_c$] and the tendency of the system to thermalize to a local equilibrium [third negative term (σ_0) on the rhs of Eq. (23)]. Also from Fig. 2(a), χ is generally increasing even when D does not change appreciably, meaning that work will be wasted in such cases where k is not large enough for synchronization to occur.

Interestingly, in the classical limit, this dynamics simplifies as

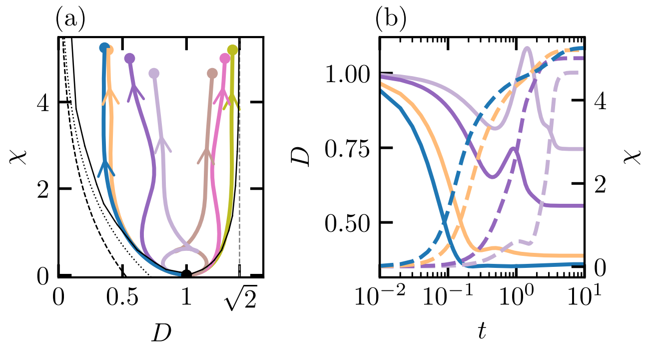

FIG. 2. Trajectories in the χ-D plane with k values (from left to right) $k = 5$, $k = 3$, $k = 1$, $k = 0.5$, $k = -0.5$, $k = -1$, $k = -3$ are shown in (a). Classical (dotted line) and quantum (dashed line) lower bounds on χ, and convex hull of 10^6 random Gaussian states (solid line). Time dependence of D (solid lines) and χ (dashed lines) with k values (solid lines bottom to top, dashed lines top to bottom) $k = 5$, $k = 3$, $k = 1$, $k = 0.5$ in (b). The frequencies of the two oscillators are $\omega_1 = 2\pi$ and $\omega_2 = 3\pi$, and they are coupled to local baths at $\beta = 1/20$ with strengths $\gamma_1 = 0.00087$ and $\gamma_2 = 0.0016$.

$$\dot{z}_1 = \left[\frac{k}{2} - i\omega_1 - \gamma_1 |z_1|^2\right] z_1 + \frac{k}{2}\Delta z - i2\sqrt{\gamma_1}\xi(t),$$

$$\dot{z}_2 = \left[\frac{k}{2} - i\omega_2 - \gamma_2 |z_2|^2\right] z_2 - \frac{k}{2}\Delta z - i2\sqrt{\gamma_2}\xi(t),$$

where $z_j = (x_j + ip_j)/\sqrt{2}$, $\Delta z = z_2 - z_1$, and $\xi(t)$ denotes an idealized delta-correlated noise process [86]. Note that the above equations describe a pair of coupled Stuart-Landau oscillators with amplitude-dependent noise [118,119]. A similar mapping of the quantum dynamics to the classical Stuart-Landau equations was discussed in Refs. [46,47,120]. However, in these cases, unlike our scenario, the most general classical Stuart-Landau description occurs only when the baths leading to the Lindbladian description are not thermal (engineered reservoirs) such that the rates do not obey local-detailed balance. We find that the classical Stuart-Landau system [121,122] displays synchronization in the regime $k \geq |\omega_2 - \omega_1|$ (see boundaries in Fig. 3 and the Supplemental Material [86]). The corresponding quantum system synchronizes for even smaller values of k beyond the strict classical boundary [86]. This presents a clear quantum advantage that extends quantum synchronization beyond classical.

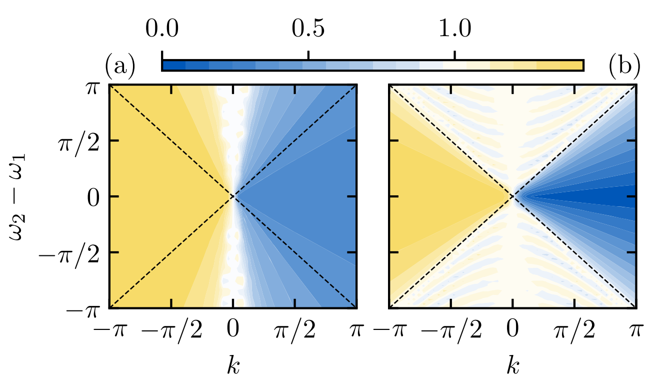

FIG. 3. Distance measure D at time $t = 10$ as a function of k and $\omega_2 - \omega_1$ for quantum (a) and classical (b) evolution. Dashed lines are $k = \pm(\omega_2 - \omega_1)$. Classical synchronous regime is for $k > |\omega_2 - \omega_1|$ (within the dashed lines), whereas for a quantum system the synchronizing regime extends well beyond the classical bounds. All other parameters are the same as Fig. 2.

Concluding remarks.—In this work, we have found the minimal amount of work required for synchronization of an arbitrary number of oscillators, as well as bounded the speed at which synchronization may occur. Our numerical results for the model of an anti-$\mathcal{PT}$-symmetric Hamiltonian serve to illustrate an experimentally realizable system where this process may occur. Our analysis allows for an information-theoretic interpretation of the synchronization process as a form of communication. This connection could be made more explicit in future work by relating our measure of synchronization to mutual information. It remains to be seen whether there are states that display a quantum advantage, in the sense of Fig. 1, and a related goal for future work is to understand the apparent quantum advantage displayed in synchronization of the dimer model (see Fig. 3).

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