Sign-Indefinite Invariants Shape Turbulent Cascades

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We highlight a noncanonical yet natural choice of variables for an efficient derivation of a kinetic equation for the energy density in nonisotropic systems, including internal gravity waves on a vertical plane, inertial, and Rossby waves. The existence of a second quadratic invariant simplifies the kinetic equation and leads to extra conservation laws for resonant interactions. We analytically determine the scaling of the radial turbulent energy spectrum. Our findings suggest the existence of an inverse energy cascade of internal gravity waves, from small to large scales, in practically relevant scenarios.

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Introduction.—Strongly dispersive waves are ubiquitous in geophysical fluid dynamics, where they occur on scales from centimeters to thousands of kilometers and contribute in an essential and intricate way to the long-term nonlinear dynamics of the climate system [1–4]. Examples include surface waves, internal inertia-gravity waves, and Rossby waves, all of which owe their existence to some combination of gravity, rotation, and curvature of the Earth. Many of these waves are far too small in scale to be resolvable numerically, making their study a pressing issue for theoretical modeling and investigation. For small-amplitude waves the methods of wave turbulence theory can play an important part in this, because they produce a closed kinetic equation for the slow evolution of the averaged spectral energy density. There has been significant progress for idealized model systems [5,6], but so far this has not yet been translated to systems of direct geophysical interest. Arguably, progress has been hampered by the extremely cumbersome form taken by the relevant equations when attempting to shoe-horn them into classical wave turbulence theory, which was formulated in canonical variables for Hamiltonian systems [7,8]. But the underlying fluid equations are noncanonical Hamiltonian systems, as is made obvious by the fact that the Euler equations are highly nonlinear yet their energy function is quadratic [9,10]. This has motivated the present work, in which we pursue a reformulation of kinetic wave theory for a number of two-dimensional fluid systems with quadratic energies based on a particular choice of noncanonical variables. The practical utility of our choice of variables, which was introduced in a different context by [11], derives from the existence of a second quadratic invariant in these systems, which, albeit not sign definite, greatly simplifies the wave interaction equations. We leverage these simplifications into a derivation of scaling laws for the isotropic component of wave spectra and we present evidence for the importance of these second invariants in shaping the overall wave spectra in certain situations. *Mutatis mutandis*, much of our analysis applies to waves in plasmas as well.

The two-dimensional Boussinesq equations restricted to a vertical xz plane can be written as

$$\begin{aligned} \Delta\psi_t + \{\psi, \Delta\psi\} &= -N^2\eta_x, \\ \eta_t + \{\psi, \eta\} &= \psi_x. \end{aligned} \tag{1}$$

Here z is the vertical and x is the horizontal coordinate with corresponding velocities w and u, ψ is a stream function such that $(\psi_x, \psi_z) = (w, -u)$ and $-\Delta\psi$ is the vorticity, η is the vertical displacement, N the constant buoyancy frequency, and $\{g, f\} = \partial_x g \partial_z f - \partial_z g \partial_x f$. The vertical buoyancy force $b = -N^2\eta$ opposes vertical displacements and derives from a consideration of potential energy in the presence of gravity and nonuniform density. It is easily checked that this system has two exact quadratic invariants: the total energy $E = \int d\mathbf{x}(-\psi\Delta\psi + N^2\eta^2)$ and the pseudomomentum $P = \int d\mathbf{x}\eta\Delta\psi$. The subtleties associated with the Hamiltonian point of view of these equations can be appreciated by investigating the origin of these conservation laws by rewriting (1) as

$$\partial_t D\phi = \mathcal{J}\frac{\delta E}{\delta(D\phi)}. \tag{2}$$

Here $\phi^T(\mathbf{x}, t) = (\psi, \eta)$, $D = \mathrm{diag}(-\Delta, N^2)$ is a Hermitian semi-positive-definite operator, and

$$\mathcal{J}(\phi) = \frac{1}{2}\begin{pmatrix} \{-\Delta\psi, \cdot\} & \{N^2\eta, \cdot\} + N^2\partial_x \\ \{N^2\eta, \cdot\} + N^2\partial_x & 0 \end{pmatrix} \tag{3}$$

is a skew-symmetric operator representing the Poisson structure. This is a noncanonical Hamiltonian system for the variables $D\phi$ based on the inner product Hamiltonian function

$$E = \langle \phi | D\phi \rangle = \int d\mathbf{x} \phi^T D\phi. \tag{4}$$

Energy conservation is then transparently linked to the time translation symmetry of $\mathcal{J}(\phi)$. The pseudomomentum can be written as

$$P = \langle D\phi | CD\phi \rangle \quad \text{with} \quad C = -\frac{1}{2N^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5}$$

Therefore, $\delta P/\delta(D\phi) = 2CD\phi$ and

$$\mathcal{J}(\phi)\frac{\delta P}{\delta(D\phi)} = -\frac{(D\phi)_x}{2}, \tag{6}$$

which ensures the invariance of P based on the x-translation symmetry of (4). This suggests interpreting P as a canonical horizontal momentum, even though it does not agree with the horizontal momentum of the fluid. Actually, $CD\phi$ is in the kernel of the nonlinear part of $\mathcal{J}$, which suggests interpreting P as a Casimir of that Poisson degenerate structure [9,12]. This degeneracy translates to gauge invariance in terms of Lagrangian coordinates [13]. Thus, the conservation of P appears Casimir-like based on the nonlinear dynamics, but momentum-like based on the linear dynamics. Calling P the "pseudomomentum" is in accordance with established usage in geophysical fluid dynamics [e.g. §4.3 in 9] and wave–mean interaction theory [2]. So, while the conceptual origins of the two conservation laws for E and P are subtle and subject to interpretation, their actual functional expressions as quadratic forms $E = \langle \phi | D\phi \rangle$ and $P = \langle D\phi | CD\phi \rangle$ are completely straightforward, and generalize easily. Following [11], we exploit this by expanding the flow in variables that diagonalize both E and P.

Wave mode expansion.—We consider a periodic domain $\mathbf{x} \in [0, L]^2$ and expand $\phi(\mathbf{x}, t) = \sum_\alpha Z_\alpha(t) g_\alpha(\mathbf{x})$ in terms of linear wave modes, where $Z_\alpha(t)$ are complex scalar wave amplitudes and the $g_\alpha(\mathbf{x})$ are eigenvector functions for the linear part of (1), i.e.,

$$-i\omega_\alpha D g_\alpha = N^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x g_\alpha. \tag{7}$$

The $g_\alpha(\mathbf{x})$ are proportional to Fourier modes $\exp(i\mathbf{k}\cdot\mathbf{x})$ with $\mathbf{k} = (k_x, k_z) \in (2\pi\mathbb{Z}/L)^2$. If $\mathbf{k} = K(\cos\theta, \sin\theta)$ then the dispersion relation is $\omega = \pm N\cos\theta$. The multi-index $\alpha = (\sigma, \mathbf{k})$ combines branch choice $\sigma = \pm 1$ and wave number $\mathbf{k}$ such that

$$\omega_\alpha = \sigma N \frac{k_x}{K} = \sigma N \cos\theta_k. \tag{8}$$

The branch choice $\sigma = \pm 1$ has physical significance because it corresponds to right-going or left-going waves, respectively. The reality of ϕ implies $Z_{(\sigma,\mathbf{k})}(t) = Z^*_{(\sigma,-\mathbf{k})}(t)$, where the star denotes complex conjugation. This holds separately within each branch. The expansion diagonalizes the energy

$$E = \sum_\alpha E_\alpha = \sum_\alpha Z_\alpha Z^*_\alpha \tag{9}$$

and yields the exact equations

$$Z_{\alpha,t} + i\omega_\alpha Z_\alpha = \sum_{\beta,\gamma} \frac{1}{2} V^{\beta\gamma}_\alpha Z^*_\beta Z^*_\gamma. \tag{10}$$

The interaction coefficients $V^{\beta\gamma}_\alpha = \langle g^*_\alpha | \mathcal{J}(g_\gamma) g_\beta + \mathcal{J}(g_\beta) g_\gamma \rangle$ are real, symmetric in upper indices and zero unless $\mathbf{k}_\alpha + \mathbf{k}_\beta + \mathbf{k}_\gamma = 0$. The expansion also diagonalizes the pseudomomentum

$$P = \sum_\alpha P_\alpha = \sum_\alpha s_\alpha Z_\alpha Z^*_\alpha, \tag{11}$$

where the horizontal slowness $s_\alpha = k_x/\omega_\alpha$. Hence $P_\alpha = k_x/\omega_\alpha E_\alpha$ has the sign of the horizontal phase (or group) velocity, which is equal to the sign of σ.

The kinetic equation.—From (9) and (10) the road to a kinetic equation is short. The modal wave energy evolves according to

$$\dot{E}_\alpha = \sum_{\beta,\gamma} V^{\beta\gamma}_\alpha \mathrm{Re}(Z^*_\alpha Z^*_\beta Z^*_\gamma). \tag{12}$$

The kinetic equation describes the evolution of $e_\alpha = \overline{E_\alpha}$, where the overbar denotes averaging over a suitable statistical ensemble. In particular, we average over random Gaussian initial conditions such that

$$\overline{Z^*_\beta(0) Z_\alpha(0)} = \delta_{\alpha\beta} e_\alpha(0) \tag{13}$$

is the only nonzero correlation. The standard assumptions and procedural steps of weak wave turbulence [5,6] then result in

$$\dot{e}_\alpha = \pi \int_{\omega_{\alpha\beta\gamma}} V^{\beta\gamma}_\alpha (V^{\alpha\gamma}_\beta e_\alpha e_\gamma + V^{\alpha\beta}_\gamma e_\beta e_\alpha + V^{\beta\gamma}_\alpha e_\beta e_\gamma). \tag{14}$$

Here the joint kinetic limits of big box and long nonlinear times, $L \to \infty$ and $t\omega \to \infty$, were taken. So the discrete sums in (10) and (12) were replaced by an integral over the resonant manifold

$$\int_{\omega_{\alpha\beta\gamma}} = \int d\beta d\gamma \delta(\omega_\alpha + \omega_\beta + \omega_\gamma)\delta(\mathbf{k}_\alpha + \mathbf{k}_\beta + \mathbf{k}_\gamma), \tag{15}$$

where $\int d\alpha = \sum_{\sigma=\pm 1} \int d\mathbf{k}$. The kinetic equation (14) is generic for three-wave interactions, but it can be greatly simplified because of the nongeneric additional conservation law for P. The conservation of E and P for every triad (even nonresonant triads) implies

$$V_\alpha^{\beta\gamma} + V_\beta^{\alpha\gamma} + V_\gamma^{\beta\alpha} = 0,$$
$$s_\alpha V_\alpha^{\beta\gamma} + s_\beta V_\beta^{\alpha\gamma} + s_\gamma V_\gamma^{\beta\alpha} = 0, \tag{16}$$

respectively [14,15]. Notably, (16) ensures conservation of E and P for any projection of (10) onto a truncated set of modes. Viewing (16) as dot products in $\mathbb{R}^3$ means that $\vec{V} = (V_\alpha^{\beta\gamma}, V_\beta^{\alpha\gamma}, V_\gamma^{\beta\alpha})$ is orthogonal to both (1, 1, 1) and $(s_\alpha, s_\beta, s_\gamma)$, which already determines the direction of $\vec{V}$ uniquely. This makes clear that any other exact quadratic invariant that is diagonalized by the linear eigenbasis $\{g_\alpha\}$ must be a linear combination of E and P. In particular, this implies that the vertical pseudomomentum based on the other slowness component cannot be exactly conserved by the full dynamics. However, the kinetic equation is restricted to the resonant manifold $\omega_\alpha + \omega_\beta + \omega_\gamma = 0$, and therefore $\vec{\omega} = (\omega_\alpha, \omega_\beta, \omega_\gamma)$ is also orthogonal to both (1, 1, 1) and $(s_\alpha, s_\beta, s_\gamma)$, the latter because $s_\alpha\omega_\alpha + s_\beta\omega_\beta + s_\gamma\omega_\gamma = \hat{\mathbf{x}}\cdot(\mathbf{k}_\alpha + \mathbf{k}_\beta + \mathbf{k}_\gamma) = 0$. This means that $\vec{V}$ and $\vec{\omega}$ are parallel to each other, i.e.,

$$(V_\alpha^{\beta\gamma}, V_\beta^{\alpha\gamma}, V_\gamma^{\beta\alpha}) = \Gamma_{\alpha\beta\gamma}(\omega_\alpha, \omega_\beta, \omega_\gamma) \tag{17}$$

for some real $\Gamma_{\alpha\beta\gamma}$ totally symmetric in its indices. For our system

$$\Gamma_{\alpha\beta\gamma} = \frac{(\sin\theta_\alpha + \sin\theta_\beta + \sin\theta_\gamma)}{\sqrt{8}}(\sigma_\alpha K_\alpha + \sigma_\beta K_\beta + \sigma_\gamma K_\gamma). \tag{18}$$

This changes (14) to

$$\dot{e}_\alpha = \pi \int_{\omega_{\alpha\beta\gamma}} \omega_\alpha \Gamma_{\alpha\beta\gamma}^2 (\omega_\alpha e_\beta e_\gamma + \omega_\beta e_\alpha e_\gamma + \omega_\gamma e_\alpha e_\beta). \tag{19}$$

Compared to (14), this is a huge simplification and resembles the structure of kinetic equations derived for canonical Hamiltonian systems [including a similar entropy function $H(t) = \int d\alpha \log e_\alpha$]. It is now apparent that on the resonant manifold additional conservation laws hold compared to the full system: any component of pseudomomentum is now conserved, so in $\mathbb{R}^d$ there are $d-1$ new conservation laws that are valid for the kinetic equation. In particular, for the two-dimensional Boussinesq system the vertical pseudomomentum $P^z = \int d\alpha k_z e_\alpha/\omega_\alpha$ is conserved by resonant interactions and hence by the kinetic equation but not by the full flow. The consequences of such additional conservation laws are not studied here.

Let us comment on the validity of the kinetic equation with respect to our initial assumption (13). The off-diagonal correlator $\overline{Z_{(+,\mathbf{k})}Z^*_{(-,\mathbf{k})}}$ has $O(\epsilon^2)$ fluctuations with frequency $2\omega_+(k)$. As long as this beating frequency does not vanish, the anomalous correlator [16] averages over time weakly to zero, so that the kinetic equation remains valid. The same holds for the correlator associated with space homogeneity $\overline{Z^2_{(\sigma,\mathbf{k})}}$. This is discussed in detail in the Supplemental Material [17] together with a detailed derivation of the kinetic equation.

Steady solutions.—The frequency (8) and the coefficients (17) are homogeneous functions of the wave numbers of degree zero and one, respectively. That is, for $\lambda > 0$,

$$\omega_{(\sigma,\lambda\mathbf{k})} = \omega_{(\sigma,\mathbf{k})}, \tag{20}$$
$$V_{\lambda\mathbf{k}_\alpha}^{\lambda\mathbf{k}_\beta\lambda\mathbf{k}_\gamma} = \lambda V_{\mathbf{k}_\alpha}^{\mathbf{k}_\beta\mathbf{k}_\gamma}. \tag{21}$$

This motivates looking for formal steady solutions to (19) that are also homogeneous in the wave numbers and can hence be written in the separable form

$$e_\alpha = e_\alpha^r(K) e_\alpha^\Omega(\theta_k) \tag{22}$$

with $e_\alpha^r = K^{-w}$ for some suitable w. It turns out that we can find possible power laws for e_α^r without having to find the specific form of e_α^Ω. This does not mean e_α^Ω is arbitrary, just that one can find e_α^r separately by relatively elementary means.

A trivial such steady solution is equipartition of energy such that $w = 0$ (and then e_α^Ω =const). This well-known solution has zero spectral flux of wave energy. Nonzero flux steady solutions are the turbulent solutions, which are in general not isotropic but can still be of the form (22). We obtain two results here. (See the Supplemental Material [17] for a detailed derivation.) The first relies on equal excitation of both branches of the dispersion relation, i.e., $e_{(-,\mathbf{k})} = e_{(+,\mathbf{k})}$ for all $\mathbf{k}$ and hence $P = 0$. Physically, this means left-going and right-going waves are equally excited. Our result here is that

$$e_\alpha^r = K^{-3} \tag{23}$$

then gives a constant nonzero energy flux in a direct cascade to larger wave numbers. The second result is based on restricting the dynamics to a single branch by setting $e_{(-,\mathbf{k})} = 0$, for example. This is a natural choice for Rossby wave dynamics, which has only a single frequency branch, but it is of course an *ad hoc* artificial constraint for internal wave dynamics [11]. In practice, this might be relevant for internal wave ensembles with very large $P > 0$, which is discussed further below. It turns out that for single-branch dynamics (23) still holds but in addition there is now a second, slightly steeper power law:

$$e_E^r(K) = K^{-3}, \qquad e_{PM}^r(K) = K^{-3.5}. \tag{24}$$

Crucially, the second power law in (24) suggests a dual cascade for internal waves, akin to the behavior familiar from two-dimensional hydrodynamic turbulence, Rossby waves, or wave turbulence based on four-wave resonances [18–20]. In such a dual cascade the pseudomomentum would go to smaller scales (direct cascade) whereas the energy would go to larger scales (inverse cascade). More details on such dual cascades are given later in the context of unidirectional wave spectra. Both power laws yield spectra of the finite capacity type [21,22], which shapes the time-dependent self-similar formation of the spectrum and might be of relevance to finite time singularity formation.

Higher dimensions.—We can generalize our results for systems of the form (10) in d dimensions, with two quadratic invariants: the energy, (9), and the pseudomomentum, (11). For $e_{(-,\mathbf{k})} = e_{(+,\mathbf{k})}$, we obtain

$$e_E^r(K) = K^{-w_E^F}, \qquad w_E^F = w_V + d - w_\omega/2. \quad (25)$$

Here w_V and w_ω are the homogeneity degrees of the interaction coefficients $V_\alpha^{\beta\gamma}$ and of the frequency ω_α. Notably, the isotropic part of the energy spectrum of an anisotropic system gets an additional contribution of $w_\omega/2$ with respect to the Kolmogorov-Zakharov power law, $w_{KZ} = d + w_V - w_\omega$, of Hamiltonian isotropic three-wave interaction systems [5,23]. Conversely, if the dynamics is restricted to one of the branches we find the additional power law

$$e_{PM}^r(K) = K^{-w_{PM}^F}, \quad w_{PM}^F = w_V + d + (1 - 2w_\omega)/2, \quad (26)$$

which corresponds to a cascade of pseudomomentum.

Returning to the complementary angular part $e_\alpha^\Omega(\theta_k)$, to find this function would require substituting the radial part (25)/(26) back into the kinetic equation and demanding that the collision integral be zero. This makes $e_\alpha^\Omega(\theta_k)$ a function of θ_k and also of the power law slope $w_{E/PM}$.

Relevance to other systems.—Our results apply to systems described by an equation of the general form (2) or equivalently (10). Rossby waves in the mid latitude beta plane, on scales smaller compared to the deformation Radius, are governed by

$$\Delta\psi_t + \{\psi, \Delta\psi\} = \beta\partial_x\psi \quad (27)$$

where ψ is the stream function on the plane, so $-\Delta\psi$ is the vorticity. x and z are the zonal and meridional position coordinates and $f = f_0 + \beta z$ is the Coriolis parameter. (27) can be written in the form (2) with $\phi = \psi, D = -\Delta$, $\mathcal{J} = \{-\Delta\psi, \cdot\}, \mathcal{L} = \beta\partial_x$. It conserves the energy (4) and a sign-definite pseudomomentum, (5) with $C = 1$, which here equals the familiar enstrophy. Rossby drift waves in plasma are described by a similar equation. The wave expansion (7) contains only one branch, $\alpha = \mathbf{k}$, so it is simply the Fourier transform. While the homogeneity degree of the interaction is the same as for internal gravity waves, the dispersion relation $\omega_k = -\beta\cos\theta/K$ is homogeneous of degree $w_\omega = -1$. Then (25) and (26) give the spectra

$$e_k^r \propto K^{-3.5}, \qquad e_k^r \propto K^{-4.5} \quad (28)$$

for inverse energy and direct enstrophy cascades, respectively. These scalings agree with the isotropic part of the steady spectra obtained by previous works [24] and are consistent with the classical notion of dual cascades [18].

The dynamics of two-dimensional inertial waves in a vertical plane with constant Coriolis parameter f is in fact governed precisely by the system we have studied in (2) after the replacement $x \leftrightarrow z, f \leftrightarrow N, \eta \leftrightarrow v_2$. ψ is the stream function on the vertical plane, v_2 is the velocity component perpendicular to the plane. Interestingly, the vertical component of pseudomomentum is then exactly conserved and is equal to the helicity of the flow. The isotropic components of energy spectra are given by (23) and (24) for the limiting cases of zero and sign-definite helicity, respectively.

We are certainly not the first to study the application of weak wave turbulence theory to internal gravity waves [14,25–29] or to Rossby waves [24,30,31]. We believe that our work is the first example of theoretical prediction for internal gravity waves in two dimensions made by wave turbulence theory and has experimental practical relevance. Previous studies considered a narrow spectral range where the homogeneous wave number component is small compared to the nonhomogeneous component. These yield spectral laws with diverging collision integrals, the divergence of the flux (19), in the case of 3D internal gravity waves and are not stable in the case of Rossby waves [32] and hence cannot be physically realized. Our solutions may represent the isotropic part of a physically relevant solution, while convergence of the collision integrals is ensured by restrictions on the complementary angular part of the energy spectrum e_k^Ω (22).

Speculations on inverse energy cascade for unidirectional internal waves.—The inverse energy cascade for Rossby waves and for internal waves can be predicted from the same argument: energy and pseudomomentum are both conserved but the spectral density of the latter differs by a factor K^n from the former, with $n = 2$ for Rossby waves and $n = 1$ for internal waves. Either way, the monotonically increasing wave number factor then implies an inverse cascade of energy and a direct cascade of pseudomomentum. Of course, this holds for internal waves only if their dynamics is artificially restricted to a single branch of horizontal propagation, as more generally the factor is σK, which is not sign-definite. This point was missed in [11], who first discussed the possibility of an inverse cascade of internal wave energy. Still, this invites speculation about a possible inverse cascade in a situation where the internal waves are almost unidirectional in the horizontal, so that one branch strongly dominates over the other. This has practical relevance for ocean dynamics. For example, strongly unidirectional internal wave fields arise naturally in the case of internal tides radiated away from isolated topography structures such as the Hawaiian ridge [33].

Specifically, split the pseudomomentum into positive and negative components via $P = P_+ + P_-$ with $P_+ \geq 0$ and $P_- \leq 0$ and consider unidirectional initial conditions that consist entirely of right-going waves such that $Z_{(-\mathbf{k})}(0) = 0$ for all $\mathbf{k}$, say. Then $P = P_+(0) > 0$ initially and in fact at all times by the exact conservation of P. First off, it follows from (12) that in this case the initial time derivatives $\dot{P}_+ = \dot{P}_- = 0$, which suggests a modicum of temporal persistence of a unidirectional wave state. Eventually nonlinear interactions will produce left-going waves of significant amplitude, but then the exact conservation of P implies that any generation of new left-going waves is accompanied by an equally strong creation of new right-going waves, which leads to a persistent state in which the right-going waves dominate such that $P_+ \gg |P_-|$ for all $t \geq 0$. This contrasts with the alternative scenario in which the initial state is unidirectional in the vertical, such that all waves have positive vertical group velocity at $t = 0$, for example. In this case upwards and downwards waves can readily equilibrate via triad interactions of the elastic scattering type [25], leading to a long-term state without directional preference for vertical propagation. Of course, this is because vertical pseudomomentum, unlike its horizontal counterpart, is not an exact invariant. In summary, one can assert that a wave state dominated by horizontally unidirectional waves will persist indefinitely in time, leading to the speculation that such a wave state is capable of an inverse energy cascade. Analogous arguments can be made in the forced-dissipative situation. Whether or not approximate single-branch wave dynamics behaves like exact single-branch wave dynamics remains an open question in wave asymptotics and turbulence theory. The extent and timescales of this approximation for the 2D Boussinesq equation remain a topic for future study.

Adding rotation.—The first step towards including both rotation and stratification in the kinetic equation for internal gravity waves (19) can be taken while retaining the two-dimensional nature of the flow, i.e., $\partial_y = 0$. This involves adding horizontal Coriolis forces to the momentum equations, which necessitates allowing for a third velocity component v_2 in the y direction. The Coriolis forces based on a constant Coriolis parameter f add only linear terms to the governing equations, so the linear part of the dynamics is described by the 3×3 operator

$$N^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \partial_x + f \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \partial_z. \tag{29}$$

The state vector is now $\phi^T = (\psi, v_2, \eta)$ and the Hermitian diagonal operator is $D = \mathrm{diag}(-\Delta, 1, N^2)$. This generalizes (1) to include the Coriolis force within the f-plane approximation, $f\hat{z} \times \mathbf{v}$. The energy is $E = \langle \phi | D\phi \rangle = \int d\mathbf{x}(-\psi\Delta\psi + v_2^2 + N^2\zeta^2)$. The dispersion relation is

$$\omega_{(\sigma,\mathbf{k})} = \sigma N \cos\theta_k \sqrt{1 + f^2 N^{-2} \tan^2\theta_k} \tag{30}$$

with $\sigma = 0, \pm$. The expansion of ϕ as in (7) then leads to the kinetic equation (14) with

$$V_\alpha^{\beta\gamma} = -\frac{\mathbf{k}_\beta \times \mathbf{k}_\gamma}{2\sqrt{8} K_\alpha K_\beta K_\gamma} \times \left(K_\gamma^2 - K_\beta^2 + f^2 s_\alpha^z (s_\gamma^z - s_\beta^z) + N^2 s_\alpha^x (s_\gamma^x - s_\beta^x) \right). \tag{31}$$

This does not include interactions among and with the zero frequency branch, also known as the balanced modes. Formally, at the limit of vanishing frequency, off-diagonal correlators should be added as well to the kinetic equation. However, weak wave turbulence closure is not expected to remain valid when balanced modes carry the dominant part of the energy [34]. Energy is an exact invariant, so the first constraint in (16) holds. As the homogeneity degree of the dispersion relation remains $w_\omega = 0$, our finding (25), suggests that rotation only modifies the angular component of the turbulent energy spectrum (22), but leaves the radial component, (23), unchanged. We note that the conservation of potential vorticity might be used to simplify the kinetic equation in the presence of both rotation and stratification. This is studied in a future work.

Conclusion.—Our work emphasizes the elegant ramifications in the theoretical description and in practice of sign-indefinite invariants, which usually do not get much attention in wave turbulence study. We show that the existence of a second quadratic invariant, simplifies the kinetic equation and leads to additional conservation laws on the resonant manifold, which to our knowledge, were previously unknown in the geophysical community. This simplification facilitates the derivation of scaling laws for the isotropic component of the turbulent wave spectra of 2D internal gravity waves and Rossby waves. We show that there are practical scenarios in which pseudomomentum conservation can drive an inverse energy cascade of internal gravity waves. On the theoretical front, our work contributes a different approach to the study of wave turbulence in nonisotropic systems dominated by three-wave interactions. This encompasses the application of noncanonical variables for deriving the kinetic equation, and using variable separation in order to find turbulent solutions of the kinetic equation.

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