Stress and Geometry for Isotropic Singularities

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We develop the mathematics needed to treat the interaction of geometry and stress at any isotropic spacetime singularity. This enables us to handle the Einstein equations at the initial singularity and characterize allowed general relativistic stress-energy tensors. Their leading behaviors are dictated by an initial hypersurface conformal embedding. We also show that an isotropic big bang determines a canonical nonsingular metric on and about the initial hypersurface as well as a cosmological time. This assigns a volume and energy to the initial point singularity.

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Introduction.—Is the causal structure of our universe singular at the big bang? This question is of physical import since, as we shall show, a well-defined causal structure at the initial big bang singularity imposes strong constraints on the matter content of the early Universe.

We perform our analysis in the context of spacetimes with isotropic singularities. As explained in [1,2], an isotropic singularity can be removed by multiplying the physical metric by some power of a suitable timelike coordinate; see Sec. III. The conformal structure can then be extended across the initial singularity, which is now described by a spacelike hypersurface. This means that all physical spacetime directions contract at the same rate when approaching the singularity. This definition need not imply any particular isometries for some choice of spatial slices. It includes the standard Friedmann-Lemaître-Robertson-Walker (FLRW) example discussed below. Isotropic singularities are relevant in light of Penrose's Weyl curvature hypothesis [3] asserting that the Weyl tensor is finite at any initial singularity even if the Ricci curvature is singular [4]. They have also been studied when subject to various underlying matter model assumptions [1,2,9–15]. Our analysis applies to generic stress-energy tensors (stress) for spacetimes with isotropic singularities.

Because the metric along an isotropic singularity is degenerate but the conformal structure is still well defined, early universe physics is dictated by the mathematics of conformally embedded hypersurfaces. Conformal submanifold embeddings are crucial to the theory of observables in the AdS/CFT correspondence [16,17]. This machinery can be fruitfully applied to cosmology. Indeed, any study of causal structures amounts to a problem in conforonal geometry. There is a well established "tractor calculus" for handling conformal geometries. Our presentation is self-contained, though many key definitions are relegated to footnotes; excellent resources for further details include [18,19].

Conformal geometry.—For simplicity we focus on generic, dimension 4 [20], causal structures given by the data of a Lorentzian conformal geometry $(M, \boldsymbol{g})$ [21], where $\boldsymbol{g}$ denotes a conformal class of metrics g with equivalence given by rescalings $\Omega^2 g \sim g$ for $0 < \Omega \in C^\infty M$. Parallelism determined by the Levi-Civita connection ∇ is a central mathematical construct of general relativity. Its conformal geometry generalization, known as the tractor connection $\boldsymbol{\nabla}$, promotes the tangent bundle TM to a "tractor bundle" $\boldsymbol{TM}$ with dimension six fibers [18,22]. Tractors are basic objects for theories incorporating local conformal transformations and diffeomorphisms [19]. They are even useful for analyzing systems that are not invariant under local Weyl symmetry. Indeed $\boldsymbol{g}$ contains a metric solving the (Λ-)vacuum Einstein equations precisely when there is a parallel tractor vector field $I \in \boldsymbol{TM}$ [18,23] [details are discussed just before Eq. (7) below], *viz*

$$\boldsymbol{\nabla} I = 0. \tag{1}$$

To incorporate Lorentz symmetry in physical theories, 3-vectors are promoted to 4-vectors. Tractors promote 4-vectors to 6-vectors to manifest conformal symmetry. Given a choice of metric $g \in \boldsymbol{g}$, a tractor I is a triple

$$I \stackrel{g}{=} \begin{pmatrix} \sigma \\ n^b \\ \rho \end{pmatrix} \stackrel{\Omega^2 g}{=} \begin{pmatrix} \Omega\sigma \\ \Omega(n + \sigma d \log \Omega)^b \\ \Omega^{-1}(\rho - \mathcal{L}_n \log \Omega - \frac{\sigma}{2}|d \log \Omega|_g^2) \end{pmatrix}, \tag{2}$$

where σ, ρ are scalars, $\mathcal{L}_n$ is the Lie derivative along the vector n, and the gauge transformation in Eq. (2) is valued in the parabolic subgroup (preserving a lightlike ray) of the spacetime conformal group SO(4,2). Tractors of any tensor rank are also well defined [18,19]. The tractor connection is defined by

$$\boldsymbol{\nabla}_a I := \begin{pmatrix} \nabla_a \boldsymbol{\sigma} - n_a \\ \nabla_a n^b + \boldsymbol{\sigma} P_a^b + \rho \delta_a^b \\ \nabla_a \rho - n_c P_a^c \end{pmatrix}. \tag{3}$$

In the above, $\boldsymbol{\sigma}$ denotes a conformal density of weight 1. A weight w conformal density [24] may be viewed as a power of a volume form, so is a section of $[(\wedge^4 TM)^2]^{(w/8)} =: \mathcal{E}M[w]$. This is a power of a tensor density, so the Levi-Civita connection is well defined acting upon it. A density $\boldsymbol{\sigma}$ may also be understood as an equivalence class of metric-function pairs $(g,\sigma) \sim (\Omega^2 g, \Omega\sigma)$. Indices are raised and lowered using $\boldsymbol{g}$ and ∇ is the Levi-Civita connection (see [19]). Also P denotes the Schouten tensor and J its trace.

The standard tractor bundle $\boldsymbol{TM}$ comes equipped with a parallel "tractor metric" h and (unlike the tangent bundle) a canonical tractor vector field $X \in \boldsymbol{TM}[1]$ [18,19,25]. Indeed, $h(I,X) = \boldsymbol{\sigma} \in \mathcal{E}M[1]$ for any I as given on the left-hand side of Eq. (2); this defines X. Moreover, when Eq. (1) holds, it turns out that the vacuum cosmological constant Λ_{vac} obeys

$$-\frac{1}{3}\Lambda_{\text{vac}} = 2\boldsymbol{\sigma}\rho + |n|_{\boldsymbol{g}}^2 = h(I,I) =: I \cdot I =: I^2.$$

To incorporate matter, we must couple stress to the right-hand side of Eq. (1), since it is a conformally covariant reformulation of Einstein's equations *in vacua*. On the other hand, when the function σ is a good coordinate for some hypersurface Σ, the local conformal embedding data $\Sigma \hookrightarrow (M, \boldsymbol{g})$ is encoded by $\boldsymbol{\nabla} I \in TM \otimes \boldsymbol{TM}$. The tractor $\boldsymbol{\nabla} I$ is a canonical conformal extension of the extrinsic curvature. It follows that there is a natural correspondence between stress and local conformal embedding data.

Isotropic singularities.—An isotropic singularity is a spacelike hypersurface Σ in a spacetime M with a degenerate physical metric $\breve{g}$ such that, for $\alpha < 0$ and any defining function τ [26],

$$g = \tau^{2\alpha} \breve{g}, \tag{4}$$

extends to a smooth metric across Σ. The degree of metric singularity for the (zero Λ) conformally flat FLRW spacetime with perfect fluid pressure to density ratio κ is

$$\alpha_{\text{FLRW}} = -\frac{2}{3\kappa + 1}.$$

Even this simplest of cosmological scenarios allows non-integer α. This parameter controls both smoothness of the physical metric $\breve{g}$ and the volume expansion rate.

Any other *bona fide* metric $g' = \Omega^2 g$ corresponds to a rescaled defining function $\tau' = \Omega^{(1/\alpha)}\tau$. Thus we may write (4) as

$$\breve{g} = \boldsymbol{\tau}^{-2\alpha} \boldsymbol{g},$$

with $\boldsymbol{\tau} \in \mathcal{E}M[1/\alpha]$, where no choice of metric in the conformal class $\boldsymbol{g}$ has been made. Indeed the causal structure of $\boldsymbol{g}$ is well defined across the initial singularity.

Our aim is to analyze the Einstein field equations

$$\breve{G} + \Lambda \breve{g} = \breve{T}, \tag{5}$$

where $\breve{T}$ is the stress of a universe with cosmological constant Λ and degenerate physical metric $\breve{g}$. For this we use maps from weight 1 scalar densities to weight 0 tractors and from tractor-valued one-forms to weight 1 symmetric trace-free tensors [19,27] (denoted by $^\circ$)

$$\boldsymbol{\mu} \overset{I}{\longmapsto} \begin{pmatrix} \boldsymbol{\mu} \\ \nabla^b \boldsymbol{\mu} \\ -\frac{1}{4}(\Box + J)\boldsymbol{\mu} \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ \mathring{x}_a^b \\ -\frac{1}{3}\nabla_b \mathring{x}_a^b \end{pmatrix} \overset{q^*}{\longmapsto} \mathring{x}_{ab}. \tag{6}$$

When $\boldsymbol{\mu}$ is nonvanishing almost everywhere, $I_{\boldsymbol{\mu}}$ is termed a *scale tractor*. For *any* weight one density $\boldsymbol{\mu}$, the definitions of $\boldsymbol{\nabla}$ and $I_{\boldsymbol{\mu}}$ in Eqs. (3) and (6) imply vanishing of the top slot of $\boldsymbol{\nabla} I_{\boldsymbol{\mu}}$. So by virtue of Eq. (2) its middle slot is a trace-free conformally covariant rank two tensor equaling $q^* \boldsymbol{\nabla} I_{\boldsymbol{\mu}}$. Remembering that $\mathring{G} = 2\mathring{P}$, it follows from Eq. (3) that $2\boldsymbol{\mu}^{-1} q^* \boldsymbol{\nabla} I_{\boldsymbol{\mu}}$ is precisely the trace-free Einstein tensor for the metric $\boldsymbol{\mu}^{-2}\boldsymbol{g}$, wherever this is defined. This explains the relationship between parallel scale tractors and Einstein metrics. Equation (5) now reads

$$q^* \boldsymbol{\nabla} I_{\boldsymbol{\tau}^\alpha} = \frac{\boldsymbol{\tau}^\alpha}{2} \mathring{\breve{T}}, \tag{7}$$

$$I_{\boldsymbol{\tau}^\alpha}^2 = \frac{1}{12} \breve{T}_a^a - \frac{1}{3}\Lambda. \tag{8}$$

Scale tractors are potentials for Einstein's equations since the derivative of I yields trace-free stress. Reference [28] shows that the Einstein–Hilbert action is the integral of I^2. Note that it already follows from Eq. (8) that the trace of the stress for a spacetime with isotropic singularity cannot vanish along Σ unless $\alpha = -1$, as the leading behavior of $I_{\boldsymbol{\tau}^\alpha}^2$ is $[\alpha(\alpha+1)/2]\boldsymbol{\tau}^{2\alpha-2}|\nabla \boldsymbol{\tau}|_{\boldsymbol{g}}^2$.

Traversing the singularity.—Spacetimes whose singularities are isotropic admit a global causal structure. Hence, even though the Einstein tensor is singular across an isotropic singularity, there are a number of well-defined geometric quantities, invariant to the structure (meaning that they are determined by the structure alone), that constrain matter. The Weyl tensor $W_{ab}{}^c{}_d$ is defined independently of any choice of $g \in \boldsymbol{g}$ but, unlike the conformally covariant Bach tensor $B_{ab} := \Box P_{ab} - \nabla^c \nabla_a P_{bc} + P^{cd} W_{acbd}$, it is not related to stress by a local differential operator. Let $\mathcal{P}$ be the conformally covariant *partially massless wave operator* defined, acting on a weight 1 trace-free symmetric tensor $\mathring{x}_{ab}$, by [29]

$$\mathcal{P}\mathring{x}_{ab} \coloneqq \Box\mathring{x}_{ab} - \nabla_c\nabla_{(a}\mathring{x}^c_{b)_\circ} - \frac{1}{3}\nabla_{(a}\nabla_{|c|}\mathring{x}^c_{b)_\circ} + W_a{}^c{}_b{}^d\mathring{x}_{cd}.$$

For any nonvanishing weight one density $\boldsymbol{\mu}$ [35],

$$\boldsymbol{\mu}B \coloneqq \mathcal{P}q^*\boldsymbol{\nabla}I_{\boldsymbol{\mu}}. \tag{9}$$

Given a causal structure $\boldsymbol{g}$, the Bach tensor is nonsingular so the above implies that the physical stress obeys a d'Alembert-type equation [36]

$$B = \frac{1}{\boldsymbol{\tau}^\alpha}\mathcal{P}\left(\frac{\boldsymbol{\tau}^\alpha}{2}\mathring{\check{T}}\right). \tag{10}$$

The Bach tensor is a natural invariant of a conformal structure [37], so the above relates causality and stress.

Singularity geometry: So far the conformal embedding data $\Sigma \hookrightarrow (M, \boldsymbol{g})$ determined by the isotropic singularity has not been used. These data determine uniquely the local asymptotics of another metric g_+, termed the *singular Yamabe metric*, whose scalar curvature obeys

$$R^{g_+} = 12 + \mathcal{O}(\sigma^4), \tag{11}$$

where σ is any defining function for Σ. Interestingly enough, the initial hypersurface Σ is a conformal infinity of g_+. An all order "singular Yamabe problem" [38–41] solution amounts to finding $\boldsymbol{\sigma} = [g, \sigma] \in \mathcal{E}M[1]$ such that $I^2_{\boldsymbol{\sigma}} = -1$ and for which $g_+ \coloneqq \boldsymbol{\sigma}^{-2}\boldsymbol{g}$. The expansion coefficient of the $\boldsymbol{\sigma}^4$ term in $I^2_{\boldsymbol{\sigma}} + 1$, along Σ, is a weight -4 conformal hypersurface invariant [41–43] equaling the variation of an energy functional E_Σ [44–46]. This energy is the anomaly in the renormalized volume of (M, g_+) [45,46] and a conformal invariant of the initial singularity

$$E_\Sigma = \int_\Sigma \mathring{K}_{ab}\mathring{F}^{ab}dV_{g_\Sigma}.$$

The above integral is over any metric in the conformal class of metrics $\boldsymbol{g}_\Sigma$ induced along Σ by $\boldsymbol{g}$. It is invariantly defined because the contraction of the trace-free extrinsic curvature $\mathring{K}$ with the *Fialkow tensor* [47,48],

$$F_{ab} \coloneqq \hat{n}^c\hat{n}^dW_{cabd} - \mathring{K}_{ac}\mathring{K}^c_b + \frac{1}{4}\mathring{K}_{cd}\mathring{K}^{cd}\bar{g}_{ab} \in \odot^2\, T^*\Sigma,$$

defines a conformal density of weight -3. The extrinsic curvatures (K, F) have respective transverse orders (1,2) and are termed second and third fundamental forms [49]. They give the first two elements in a sequence of trace-free conformal hypersurface invariants defined along Σ and termed *conformal fundamental forms* [50]. These are conformally invariant obstructions to the problem of finding an asymptotically de Sitter (dS) metric with conformal infinity Σ. They probe derivatives of $\boldsymbol{g}$ off of Σ in the direction of the (future-pointing) timelike unit normal $\hat{n} \in TM[-1]|_\Sigma$.

The extrinsic curvature K measures the difference between the Levi-Civita connections ∇ and $\overline{\nabla}$ of M and Σ respectively, while the Fialkow tensor measures that of the respective tractor connections $\boldsymbol{\nabla}$ and $\bar{\boldsymbol{\nabla}}$ of $\boldsymbol{g}$ and $\boldsymbol{g}_\Sigma$ [48].

The *fourth conformal fundamental form* [50,51] takes three normal derivatives of $\boldsymbol{g}$ [52],

$$\mathring{L}_{ab} \coloneqq (\hat{n}^cC_{c(ab)})^\top + H\hat{n}^c\hat{n}^dW_{acbd} - \overline{\nabla}^c(\hat{n}^dW_{d(ab)c})^\top \in \odot^2\, T^*\Sigma[-1].$$

Here $C_{abc} \coloneqq 2\nabla_{[a}P_{b]c}$ is the Cotton tensor and $\odot$ denotes the trace-free symmetric product of one forms. The tensor $\mathring{L}$ is distinguished in the context of dS_4 metrics, because rather than obstructing solutions, it extracts the second piece of boundary data (the first being $\boldsymbol{g}_\Sigma$).

The locally determined singular Yamabe asymptotics of (11) terminate at order four, so the fourth conformal fundamental form $\mathring{L}$ is the last tensor determined this way. Before using the geometric triple $(\mathring{K}, \mathring{F}, \mathring{L})$ to constrain $\mathring{\check{T}}$, we study further (pseudo-)Riemannian data determined by the isotropic singularity.

The big bang metric: Remarkably there is a canonical Riemannian metric along the big bang hypersurface Σ. It is constructed from the solution $\boldsymbol{\sigma}$ to the singular Yamabe problem: The weight 1 density $\boldsymbol{\gamma}$ defined by

$$\boldsymbol{\gamma}^{\frac{1}{\alpha}-1} \coloneqq \frac{\boldsymbol{\tau}}{\boldsymbol{\sigma}} \tag{12}$$

is nowhere vanishing by our smoothness assumptions and therefore defines a Lorentzian metric

$$g_\gamma \coloneqq \boldsymbol{\gamma}^{-1}\boldsymbol{g},$$

and, in particular, a Riemannian metric g_Σ on the spacelike isotropic singularity Σ [53]. Thus the Riemannian three manifold (Σ, g_Σ) is an invariant of the big bang, as is its volume $V_\Sigma = \int_\Sigma dV_{g_\Sigma}$.

Because the "big bang metric" g_γ is an everywhere smooth element of the conformal class $\boldsymbol{g}$, it determines the triple of conformal fundamental forms $(\mathring{K}, \mathring{F}, \mathring{L})$ by the formulæ above. Importantly the metric g_γ also furnishes the early universe with a canonical cosmological time coordinate

$$t \coloneqq \frac{\boldsymbol{\sigma}}{\boldsymbol{\gamma}}$$

depending on data of the isotropic singularity alone. It is very useful for analyzing physical stress at a big bang singularity.

Big bang stress.—The behavior of stress at an isotropic singularity can be studied using potentials I_{τ^α}, I_{σ}, and I_{γ}. By virtue of Eq. (12) and the definition of a scale tractor in Eq. (6), these must be related:

$$I_{\tau^\alpha} = t^{\alpha-2}\left[-\frac{\alpha(\alpha-1)}{4\gamma}I_{\sigma}^2 X + t\left(\alpha I_{\sigma} + \frac{\alpha(\alpha-1)}{2\gamma}I_{\sigma}.I_{\gamma}X\right) + t^2\left((1-\alpha)I_{\gamma} - \frac{\alpha(\alpha-1)}{4\gamma}I_{\gamma}^2 X\right)\right]. \quad (13)$$

This is the fundamental equation governing stress at an isotropic singularity, all quantities above are determined by the basic geometric data $\check{g}$ or, equivalently, $(\boldsymbol{g}, \boldsymbol{\tau})$.

Trace of stress: The square of a scale tractor measures scalar curvature or trace of stress [see Eq. (8)] so Eq. (13) implies

$$\frac{1}{4}\check{T}_a^a - \Lambda = \frac{3}{2}t^{2(\alpha-1)}\left[-\alpha(\alpha+1) + 2t\alpha(\alpha-1)H^{\rm ext} - \frac{t^2}{12}(\alpha-1)(\alpha-2)R^{g_\gamma} + \mathcal{O}(t^4)\right]. \quad (14)$$

In the above $H^{\rm ext} \coloneqq -I_{\sigma}\cdot I_{\gamma}$ canonically extends the mean curvature of $\Sigma \hookrightarrow (M, g_\gamma)$.

As discussed earlier, solving Einstein's equations can be broken into two steps, (i) solve for a causal structure and (ii) determine which metric in the corresponding conformal class is physical. Hence we pose the question: Given only the trace of stress and causal structure for a spacetime with isotropic singularity, can we recover the physical metric $\check{g}$? Remarkably there exists a "solution generating algebra" that addresses this question: The operator I acting on $\boldsymbol{\sigma}$ (of Sec. III) is both conformally invariant and second order. It is an example of a more general *Thomas D-operator* mapping tractors to tractors [18,25]. Given the data of a weight $w' \neq 0, -1$ density $\boldsymbol{\mu}$, this yields a conformally invariant "d'Alembert-Robin" operator [54]

$$\boldsymbol{L}_{\boldsymbol{\mu}} \coloneqq -w'\boldsymbol{\mu}(\Box + wJ) + 2(w+1)\nabla_a\boldsymbol{\mu}\nabla^a - \frac{w(w+1)}{w'+1}(\Box\boldsymbol{\mu} + w'J\boldsymbol{\mu}), \quad (15)$$

mapping weight w densities to weight $w + w' - 2$ densities. When $g_{\boldsymbol{\mu}} \coloneqq \boldsymbol{\mu}^{-(2/w')}\boldsymbol{g}$ is a metric, this gives a d'Alembert operator $\Box^{g_\mu} + [w(w+w'+2)/6(w'+1)]R^{g_\mu}$. Specializing $\boldsymbol{\mu}$ to the singular Yamabe defining density $\boldsymbol{\sigma}$, the operator $\boldsymbol{L}_{\boldsymbol{\sigma}}$ yields a conformally invariant, Robin-type, boundary operator [57]

$$\delta_{\rm R} \overset{\Sigma}{\coloneqq} \nabla_{\hat{n}} - wH.$$

The crucial point now is that, calling $\mathcal{S}_{\boldsymbol{\mu}} \coloneqq \boldsymbol{L}_{\boldsymbol{\mu}}\boldsymbol{\mu}$, there is an $\mathfrak{sl}(2) = \langle x, [x,y], y\rangle$ algebra generated by

$$(x, y) \coloneqq \left(\boldsymbol{\mu}, -\frac{2(w'+1)}{w'\mathcal{S}_{\boldsymbol{\mu}}}\boldsymbol{L}_{\boldsymbol{\mu}}\right).$$

Since Eq. (8) can be rewritten as $\boldsymbol{L}_{\boldsymbol{\tau}^\alpha}\boldsymbol{\tau}^\alpha = \frac{1}{3}\check{T}_a^a - \frac{4}{3}\Lambda$, its formal asymptotics can be determined iteratively using the solution generating $\mathfrak{sl}(2)$ algebra, cf. [56].

Conformal fundamental forms and stress: We now analyze the trace-free part of the matter coupled Einstein system in terms of conformal embedding geometry. Equation (7) implies that we must study the tractor gradient of Eq. (13) relating the various scale tractors. Acting with q^* [see Eq. (6)] and multiplying by $t^{-\alpha}$ gives

$$t^{-\alpha}q^*\boldsymbol{\nabla}I_{\tau^\alpha} = \frac{\alpha(\alpha-1)}{t^2}\boldsymbol{\gamma}dt \odot dt + \frac{\alpha}{t}q^*\boldsymbol{\nabla}I_{\sigma} + (1-\alpha)q^*\boldsymbol{\nabla}I_{\gamma}. \quad (16)$$

Multiplying by an overall factor $2/\boldsymbol{\gamma}$, each (trace-free) term above has a physical interpretation: The left-hand side is the physical stress $\mathring{\check{T}}$. The first summand is proportional to the stress of a perfect fluid with covelocity dt. The second is α times the stress of the singular Yamabe metric. It captures the embedding data. The last is $1-\alpha$ times the stress of the big bang metric g_γ. Hence, we learn the asymptotics of trace-free stress $\mathring{\check{T}}$ for all spacetimes with an isotropic singularity:

$$\mathring{\check{T}} = \frac{\alpha(\alpha-1)\mathring{T}_{\rm fluid}}{t^2} + \frac{\alpha\mathring{T}_{\rm Bach}}{t} - (\alpha-1)\mathring{T}_{\rm Big\ Bang}, \quad (17)$$

where $\mathring{T}_{\rm fluid} \coloneqq 2dt \odot dt$. Note that Eq. (9) implies

$$\mathcal{P}q^*\boldsymbol{\nabla}I_{\sigma} = \boldsymbol{\sigma}B \overset{\Sigma}{=} 0, \quad (18)$$

so the (transverse order 2) partially massless operator acting on $(\boldsymbol{\gamma}/2)\mathring{T}_{\rm Bach}$ returns $\boldsymbol{\sigma}B$.

As advertised, Eq. (17) characterizes allowed stress at an isotropic singularity. As we next show, the coefficients of terms that diverge as $t \to 0$ are local invariants of the boundary.

We want to study the first 4 orders of the early time ($t \sim 0$) asymptotics of physical stress. Both the fluid and big bang terms in Eq. (17) are completely determined to this order so we focus on the Bach term. Conformally invariant transverse jets of $q^*\boldsymbol{\nabla}I_{\sigma}$ generate the second and third but not fourth conformal fundamental forms [see Eq. (18)]. There is a notion of a fifth fundamental form, *viz* the projected Bach tensor $B^\top|_\Sigma$. However the Bach-to-stress Eq. (10) determines the conformal structure $\boldsymbol{g}$ given initial data of the first through fourth fundamental forms, so we focus on these.

Geometry	Stress
$\hat{n} \circledcirc \hat{n}$	$\frac{\gamma^2}{2}\mathring{T}_{\text{fluid}}\big\|_{\Sigma}$
$\mathring{K}$	$\frac{\gamma}{2}\mathring{T}_{\text{Bach}}\big\|_{\Sigma}$
$\mathring{F}$	$\delta_{\text{R}}\left(\frac{\gamma}{2}\mathring{T}_{\text{Bach}}\right)$
$\mathring{L}$	$\gamma^{-1}\delta_{\text{R}}\left(\frac{\gamma}{2}\mathring{T}_{\text{Big Bang}}\right) - \gamma^{-1}\delta^{(2)}\gamma$

FIG. 1. Conformal fundamental forms related to stress.

First note the second fundamental form here obeys

$$\mathring{K} = q^*\nabla I_{\sigma}|_{\Sigma} = \frac{\gamma}{2}\mathring{T}_{\text{Bach}}|_{\Sigma}.$$

To study the next order term, we use the tractor analog of the d'Alembert-Robin operator $\boldsymbol{L}_{\sigma}$ of Eq. (15) to make a transverse order 1 operator [50], again called δ_{R},

$$\circledcirc T^*M[w] \ni \mathring{x}_{ab} \overset{\delta_{\text{R}}}{\mapsto} [(\nabla_{\hat{n}} + (2-w)H)\mathring{x}_{ab} + \frac{2}{w-3}\overline{\nabla}_{(a}\mathring{x}^{\top}_{\hat{n}b)}]^{\top,\circ} \in \circledcirc T^*\Sigma[w-1].$$

The trace-free Fialkow tensor is then

$$\mathring{F} = \delta_{\text{R}} q^*\nabla I_{\sigma} = \delta_{\text{R}}\left(\frac{\gamma}{2}\mathring{T}_{\text{Bach}}\right).$$

Because we cannot extract $\mathring{L}$ from a conformally invariant second normal derivative of $q^*\nabla I_{\sigma}$ to relate the fourth fundamental form to stress, we instead consider one normal derivative of big bang stress $\mathring{T}_{\text{Big Bang}}$. For this we employ the identity [58]

$$\delta_{\text{R}} q^*\nabla I_{\gamma} = \gamma\mathring{L} + \delta^{(2)}\gamma.$$

This yields the last line of Fig. 1 summarizing the relations between geometry and stress.

Example: Poincaré-Einstein conformal cyclic cosmology.—Models where the present universe is seeded by pre-big bang data [60–63], dovetail with the above results. One approach [64] employs an asymptotically dS pre-big bang metric $\hat{g}$ and a physical metric $\check{g}$ with isotropic singularity:

$$\hat{g} = \frac{-d\hat{t}^2 + \hat{h}(\hat{t})}{\hat{t}^2}, \qquad \check{g} = \check{t}^{-2\alpha}(-d\check{t}^2 + \check{h}(\check{t})).$$

The conformal infinity or initial singularity hypersurface Σ is at $\hat{t} = 0 = \check{t}$. The pre-big bang spatial metric $\hat{h}$ is defined by a Fefferman-Graham-type expansion [65] about the conformal infinity of $\hat{g}$ obtained by solving Einstein's equations with nonvanishing stress for suitable late time $\hat{t} \to 0_-$ matter content. Conformal fundamental forms are covariant analogs of Fefferman-Graham expansion coefficients [50] and are determined by $\hat{h}(\hat{t})$. They can be matched [51] to those of the big bang model and thus its stress. Schematically,

$$\hat{\mathring{T}} \mapsto \text{conformal fundamental forms} \mapsto \check{\mathring{T}}.$$

Just as for stellar models where interior and exterior solutions are matched using fundamental forms [66–68], "cyclic cosmological matching" of conformal structures is via conformal fundamental forms.

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