Unveiling the Stabilizer Group of a Matrix Product State

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We present a novel classical algorithm designed to learn the stabilizer group—namely, the group of Pauli strings for which a state is a ± 1 eigenvector—of a given matrix product state (MPS). The algorithm is based on a clever and theoretically grounded biased sampling in the Pauli (or Bell) basis. Its output is a set of independent stabilizer generators whose total number is directly associated with the stabilizer nullity, notably a well-established nonstabilizer monotone. We benchmark our method on T-doped states randomly scrambled via Clifford unitary dynamics, demonstrating very accurate estimates up to highly entangled MPS with bond dimension $\chi \sim 10^3$. Our method, thanks to a very favorable scaling $\mathcal{O}(\chi^3)$, represents the first effective approach to obtain a genuine magic monotone for MPS, enabling systematic investigations of quantum many-body physics out of equilibrium.

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Introduction.—Quantum states of many interacting particles (or qubits) have in general a very high degree of complexity, due to exponential vastness of the Hilbert space [1,2]. Understanding which states can be efficiently simulated using reasonable classical resources, i.e., polynomial in the number N of particles, is crucial for identifying potential areas of quantum advantage [1,3–5] or improving our knowledge of out-of-equilibrium states of matter [6–9]. There are (at least) two known general classes of states falling into this scenario.

First, states with sufficiently low amount of quantum correlations between the constituencies, i.e., low entanglement, can be simulated by means of tensor etworks (TNs) [10–12]. TNs, including matrix product states (MPSs), represent quantum wave functions using tensors, where indices correspond to physical variables (like spin) and auxiliary fictitious variables. The latter are always contracted (summed over) and encode entanglement. In 1D, MPSs can simulate states whose entanglement does not scale extensively with N [13–15].

Additionally, another framework for simulating specific quantum states is linked to the Clifford group's structure [5,16–20]. This group consists of unitary transformations mapping the group of strings of Pauli matrices over N qubits (Pauli group) to itself under conjugation. Any state $|\psi\rangle$ is associated with an Abelian subgroup of the Pauli group, known as stabilizer group and denoted $G_{\mathcal{S}}(|\psi\rangle)$. This is generated by k_ψ commuting Pauli strings $\boldsymbol{\sigma}$ which stabilize the state, i.e., such that $\boldsymbol{\sigma}|\psi\rangle = \pm|\psi\rangle$. When $k_\psi = N$, the state is dubbed a stabilizer and is uniquely identified by the generators of $G_{\mathcal{S}}(|\psi\rangle)$ [5]. Stabilizers can be equivalently characterized as states obtained by applying Clifford transformations to the computational basis state $|0\rangle^{\otimes N}$, and accordingly can encode arbitrary amounts of entanglement. Nevertheless, they can be simulated classically using the stabilizer formalism. Indeed, since 2 classical bits can encode the 4 Pauli matrices, one can store the generators of $G_{\mathcal{S}}(|\psi\rangle)$ in an $N \times 2N$ tableau. Operations such as evaluating expectation values of Pauli operators or applying Clifford unitaries can be performed efficiently by updating the tableau at cost $\mathcal{O}(N^2)$ [19]. Yet, for quantum computational universality, non-Clifford unitary transformations are needed, often leading to exponential scaling in the tableau algorithm's complexity. The amount of non-Clifford resources needed to prepare a state, known as nonstabilizerness, is considered one of the veritable resources of the quantum realm [21–23].

Interestingly, only a few works have explored interconnections between these two approaches. Recently, stabilizer Rényi entropies (SREs) gained significant interest for quantifying nonstabilizerness [23–29], with the advantage of being amenable to experimental measurements [30,31]. Ways of evaluating SREs for MPSs have been discussed in Refs. [32–35]. For qubits [36] and pure states the monotonicity of SRE with Rényi index β has been recently established for $\beta \geq 2$ [37]. However, a violation of monotonicity for $0 \leq \beta < 2$ is known [34]. Thus, the question of how to extract a veritable measure of magic from an MPS remains open. In this Letter, we consider the task of learning the stabilizer group $G_{\mathcal{S}}(|\psi\rangle)$ of an MPS $|\psi\rangle$. The goal is achieved by employing a new type of sampling in the Pauli basis, that is intentionally biased to favor the extraction strings belonging to the stabilizer group. To ensure a comprehensive mapping of the entire group

$G_{\mathcal{S}}(|\psi\rangle)$, we iterate the sampling over modified states obtained with Clifford transformations from the original MPS, the computational cost for iteration being $\mathcal{O}(N\chi^3)$, where χ is the MPS bond dimension. Our algorithm outputs stabilizer generators and estimates the state's stabilizer dimension. Benchmarks show that this estimate is consistently accurate; i.e., one has a high probability of correctly learning all the generators of $G_{\mathcal{S}}(|\psi\rangle)$.

The task we consider is of crucial importance, in fact our novel algorithm is the first known method to obtain a genuine magic monotone for MPS with reasonable computational resources. Consequently, it can pave the way to a systematic numerical investigation of the nonstabilizerness of quantum many-body states. In addition, one could in principle exploit the knowledge of the stabilizer group of the MPS to reduce its computational complexity. For instance, given the k_ψ generators of $G_{\mathcal{S}}(|\psi\rangle)$, one can efficiently find a Clifford unitary $U_{\mathcal{C}}$ serving as (partial) disentangler: $U_{\mathcal{C}}|\psi\rangle = |0\rangle^{\otimes k_\psi}|\tilde{\psi}\rangle$ [5]. This could result into powerful hybrid MPS stabilizer techniques.

Preliminaries.—We consider a system consisting of N qubits. We identify the Pauli matrices by $\{\sigma^\alpha\}_{\alpha=0}^3$, with $\sigma^0 = \mathbb{1}$, and with $\boldsymbol{\sigma} = \prod_{j=1}^N \sigma_j \in \mathcal{P}_N$ a generic N-qubits Pauli string where $\mathcal{P}_N = \{\sigma^0, \sigma^1, \sigma^2, \sigma^3\}^{\otimes N}$. For a pure normalized state $\rho = |\psi\rangle\langle\psi|$, we define the stabilizer group $G_{\mathcal{S}}(|\psi\rangle)$ as the set of Pauli strings for which $|\psi\rangle$ is an eigenstate; i.e., $G_{\mathcal{S}}(|\psi\rangle) = \{\boldsymbol{\sigma}\,\text{s.t.}\,\boldsymbol{\sigma}|\psi\rangle = \pm|\psi\rangle\}$. $G_{\mathcal{S}}$ is an Abelian subgroup of the N qubits Pauli group [5].

The stabilizer dimension k_ψ of $|\psi\rangle$ is the number of independent (commuting) generators of $G_{\mathcal{S}}$. Equivalently, $k_\psi \equiv \log_2|G_{\mathcal{S}}(|\psi\rangle)|$, where $|\cdot|$ represents the cardinality of the set. The stabilizer nullity is defined as $\nu_\psi = N - k_\psi$ and is a genuine magic monotone since it is nonincreasing under any stabilizer operations, such as Clifford unitaries or measurements of Pauli operators [38].

By definition, $k_\psi = N$ for stabilizer states, whereas $k_\psi \geq N - t$ for t-doped stabilizer state (see Lemma 1 in Supplemental Material [39]) [40]. These are states obtained from the computational basis state $|0\rangle^{\otimes N}$ through the application of a circuit consisting of Clifford gates and at most t single-qubit non-Clifford T gates ($t = 0$ for stabilizers). T gate is defined as $\mathrm{T} = \mathrm{diag}(1, e^{i\pi/4})$, and together with Clifford gates {H,S,CNOT} form a universal set of gates. The number t of T gates required to synthesize a state $|\psi\rangle$ is lower bounded by ν_ψ [41].

In Ref. [42], Montanaro introduced a learning procedure for stabilizer states which is based on Bell sampling, i.e., joint measurements on two copies of the state in the Bell basis. Such measurements correspond to sampling Pauli strings $\boldsymbol{\sigma} \in \mathcal{P}_N$ with probability $\Pi_\rho(\boldsymbol{\sigma}) = (1/2^N)\mathrm{Tr}[\rho\boldsymbol{\sigma}]^2$, and we will also refer to it as Pauli sampling. Montanaro showed that for stabilizer states the generators of $G_{\mathcal{S}}(|\psi\rangle)$ can be learned with $\mathcal{O}(N)$ samples from Π_ρ [42], resulting in an exponential speedup for stabilizer states compared to tomographic methods [43]. The question of whether and how Montanaro's method can be expanded to learn t-doped states is a significant subject of research [40,44,45].

In Ref. [44] the authors employ in turn the Bell sampling and notice that the extraction of $k_\psi + N$ random Pauli operators from $G_{\mathcal{S}}(|\psi\rangle)$ is sufficient to determine a generator set with failure probability of at most 2^{-N}. However, for a t-doped state the probability of obtaining a string $\boldsymbol{\sigma} \in G_{\mathcal{S}}(|\psi\rangle)$ is $p(\boldsymbol{\sigma}\in G_{\mathcal{S}}) = |G_{\mathcal{S}}(|\psi\rangle)|\cdot(1/2^N) = 2^{k_\psi - N} \geq 2^{-t}$, since $\Pi_\rho(\boldsymbol{\sigma}) = 1/2^N$ for any stabilizer string. This result shows that, in general, the probability of successfully finding a stabilizer string decreases exponentially with t. Hence, the approach in Ref. [44] is feasible only when $t = \mathcal{O}(\log N)$.

MPS stabilizer sampling.—We consider a pure state $|\psi\rangle$ represented in the MPS form [10–12] $|\psi\rangle = \sum_{s_1,s_2,\ldots,s_N} \mathbb{A}_1^{s_1}\mathbb{A}_2^{s_2}\cdots\mathbb{A}_N^{s_N}|s_1, s_2, \ldots, s_N\rangle$, with $\mathbb{A}_j^{s_j}$ being $\chi_{j-1}\times\chi_j$ matrices, except at the left (right) boundary where $\mathbb{A}_1^{s_1}$ ($\mathbb{A}_N^{s_N}$) is a $1\times\chi_1$ ($\chi_{N-1}\times 1$) row (column) vector. Here $|s_j\rangle \in \{|0\rangle, |1\rangle\}$ is the local computational basis. Without loss of generality, the state is assumed right-normalized, namely, $\sum_{s_j}\mathbb{A}_j^{s_j}(\mathbb{A}_j^{s_j})^\dagger = \mathbb{1}$.

As shown for the first time in Ref. [32], given the MPS $|\psi\rangle$ one can efficiently achieve a perfect Pauli sampling from the probability distribution $\Pi_\rho(\boldsymbol{\sigma})$ at computational cost $\mathcal{O}(N\chi^3)$, with $\chi = \max_i\chi_i$. Such a sampling is achieved by exploiting the decomposition $\Pi_\rho(\boldsymbol{\sigma}) = \pi_\rho(\sigma_1)\pi_\rho(\sigma_2|\sigma_1)\ldots\pi_\rho(\sigma_N|\sigma_1\cdots\sigma_{N-1})$, where $\pi_\rho(\sigma_i|\sigma_1\cdots\sigma_{i-1}) = \pi_\rho(\sigma_1\cdots\sigma_i)/\pi_\rho(\sigma_1\cdots\sigma_{i-1})$ is the probability of Pauli matrix σ_i appearing at position i given that $\sigma_1\ldots\sigma_{i-1}$ have been extracted at positions $1\ldots i-1$, regardless of the occurrences in the remaining part of the system (i.e., marginalizing over Pauli substrings on qubits $i+1\ldots N$). Perfect sampling operates going through qubits $i = 1, 2\ldots N$ in a sweep and sampling each local Pauli matrix according to the conditional probabilities $\pi_\rho(\sigma_i|\sigma_1\cdots\sigma_{i-1})$ [32,46,47].

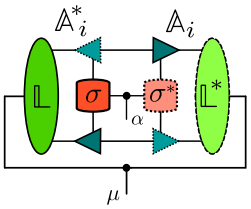
Here, we aim to identify the stabilizer group $G_{\mathcal{S}}(|\psi\rangle)$ of the MPS. In principle, one could perform a perfect sampling of Pauli strings $\boldsymbol{\sigma}$ with the hope of sampling stabilizer strings. However, as outlined before, the probability of sampling a stabilizer string decreases exponentially as k_ψ decrease. Nevertheless, in an MPS simulation a perfect sampling is not the sole option, and one can alternatively attempt to bias the extraction of stabilizer strings.

With this spirit, we introduce a novel sampling strategy. In this approach, at a generic step i of the sweep, a certain number K of substrings $\{\boldsymbol{\sigma}_{[1,i]}^\mu\}_{\mu=1}^K$ are stored [here and in the following we use $\boldsymbol{\sigma}_{[1,i]}$ as short form for $(\sigma_1\ldots\sigma_i)$]. We also store the list of corresponding partial probabilities $\{\pi_\rho(\boldsymbol{\sigma}_{[1,i]}^\mu)\}_{\mu=1}^K$, where

$$\pi_\rho(\sigma_1\cdots\sigma_i)=\sum_{\boldsymbol{\sigma}\in\mathcal{P}_{N-i}}\frac{1}{2^N}\mathrm{Tr}[\rho\sigma_1\cdots\sigma_i\boldsymbol{\sigma}]^2. \tag{1}$$

Ideally, one would like to keep track of all possible substrings, thereby enabling the identification of those that meet the stabilizer condition $\Pi_\rho(\boldsymbol{\sigma}^\mu)=\pi_\rho(\boldsymbol{\sigma}^\mu_{[1,N]})=1/2^N$ at $i=N$. Yet, this is feasible only until the number of stored substrings, which is 4^i, remains under control. In practice, one has to find effective ways of discarding certain substrings to ensure that their total number remains within a predefined maximum number $\mathcal{N}$. Naturally, the goal is to keep in memory only those substrings that have a higher likelihood of resulting into stabilizer strings at the end of the sweep. To this purpose, we adopt the following two strategies. (i) We notice that for any stabilizer string $\boldsymbol{\sigma}\in G_\mathcal{S}(|\psi\rangle)$, the partial probability at site i is lower bounded by $1/(2^i\chi_i)$; i.e., $\pi_\rho(\boldsymbol{\sigma}_{[1,i]})\geq 1/(2^i\chi_i)$ (see Lemma 3 in Supplemental Material [39]). Accordingly, one can discard all stored substrings for which $2^i\chi_i\pi_\rho(\boldsymbol{\sigma}_{[1,i]})<1$. (ii) When K exceed $\mathcal{N}$, one can simply sort the probabilities $\pi_\rho(\boldsymbol{\sigma}^\mu_{[1,i]})$ in descending order and select the substrings corresponding to the highest $\mathcal{N}$ values [48]. Indeed, these are the substrings with the highest likelihood to maximize the final probability $\Pi_\rho(\boldsymbol{\sigma})$ at the end of the sweep. Previous points establish a straightforward method for conducting a sampling process with an enhanced ability to generate Pauli stabilizer strings. At a generic step i, one has to compute the conditional probabilities $\pi(\alpha|\mu)=\pi_\rho(\sigma^\alpha|\boldsymbol{\sigma}^\mu_{[1,i-1]})$ $(\alpha\in\{0,1,2,3\}$, $\mu\in\{0,1,\ldots K\})$ and, if their total number $4K$ exceed $\mathcal{N}$, rules (i) and (ii) are used to choose $\mathcal{N}$ optimal substrings, denoted by indices $(\alpha_\star,\mu_\star)$, with all other alternatives being discarded. The selected indices are then effectively merged at the end of the step. The $\pi(\alpha|\mu)$ are obtained through the tensor contraction

$$\pi_\rho(\sigma^\alpha|\boldsymbol{\sigma}^\mu_{[1,i-1]})=\frac{1}{2}$$



(2)

using a set of environment matrices $\mathbb{L}^\mu$ (as in Ref. [32]). These serve to encode information regarding samples collected from previously visited sites, and are updated after the selection of optimal substrings as $\mathbb{L}^\mu\to 1/(2\pi_\rho(\sigma^{\alpha_\star}|\boldsymbol{\sigma}^{\mu_\star}_{[1,i-1]}))^{1/2}\cdot\sum_{s',s}(\sigma^{\alpha_\star})_{s's}(\mathbb{A}_i^{s'})^\dagger\mathbb{L}^{\mu_\star}\mathbb{A}_i^s$. The prefactor ensures a correct normalization; i.e., $\mathrm{Tr}[\mathbb{L}^\mu(\mathbb{L}^\mu)^\dagger]=1$ at each step. Initially, K is set to 1 and $\mathbb{L}^\mu=(1)$. We summarize the full stabilizer sampling recipe in Algorithm 1. The output is a set of $K\leq\mathcal{N}$ stabilizer strings. In order to find the generators of $G_\mathcal{S}(|\psi\rangle)$, one has to extract a minimal set of independent Pauli generators out of them. This task can be conveniently performed by applying Gaussian elimination on the tableau matrix obtained with the replacements $\sigma^0\to(0,0)$, $\sigma^1\to(1,0)$, $\sigma^2\to(1,1)$, $\sigma^3\to(0,1)$ from the list of samples [5]. This tableau has shape $K\times 2N$ and its reduction can be performed at cost $\mathcal{O}(\mathcal{N}N^2)$, if $N<\mathcal{N}$. In practice, these operations are extremely fast since they involve only bitwise operations. The final result is a set of independent generators of $G_\mathcal{S}(|\psi\rangle)$.

Algorithm 1. Stabilizer sampling from MPS.

Input: a right-normalized MPS $|\psi\rangle$ of size N
Initialize: $K=1$, $\{\mathbb{L}^\mu\}_{\mu=1}^K=\{(1)\}_{\mu=1}^K$, $\{\Pi^\mu\}_{\mu=1}^K=\{1\}_{\mu=1}^K$.
for $(i=1,\ i=N,\ i++)$ **do**
  Compute $\pi(\alpha|\mu)=\pi_\rho(\sigma^\alpha|\boldsymbol{\sigma}^\mu_{[1,i-1]})$
  for $\alpha\in\{0,1,2,3\}$ and $\mu\in\{1,2\ldots K\}$
  Select the $(\alpha_\star,\mu_\star)$ s.t. $\pi(\alpha_\star|\mu_\star)\Pi^{\mu_\star}\geq 1/(2^i\chi_i)$
  Set $K=\min(|\{(\alpha_\star,\mu_\star)\}|,\mathcal{N})$
  Select K indices $(\alpha_\star,\mu_\star)$ corresponding to
  largest values of $\pi(\alpha|\mu)\Pi^\mu$, discard the others
  for $(\mu=1,\ \mu=K,\ \mu++)$ **do**
    Set $\{\boldsymbol{\sigma}^\mu_{[1,i]}\}=\{(\boldsymbol{\sigma}^{\mu_\star}_{[1,i-1]},\sigma^{\alpha_\star})\}$
    Update $\{\Pi^\mu\}\to,\{\pi(\alpha_\star|\mu_\star)\Pi^{\mu_\star}\}$ and $\{\mathbb{L}^\mu\}$
  end for
end for
Output: $K\leq\mathcal{N}$ stabilizer Pauli string $\{\boldsymbol{\sigma}^\mu\}_{\mu=1}^K$

Iterations over modified states.—While these strategies are already effective in ensuring a favorable probability of sampling $G_\mathcal{S}(|\psi\rangle)$, this can be further increased. Firstly, after completing the sampling sweep from $i=1$ to $i=N$, one can repeat it in the reverse direction, i.e., from $i=N$ to $i=1$. Before the reversed sweep one has to put the MPS in the left-normalized gauge [11]. Secondly, one can repeat the sampling with modified states $|\psi'\rangle=U_\mathcal{C}|\psi\rangle$, where $U_\mathcal{C}\in\mathcal{C}_N$. Indeed, the dimension of the stabilizer group is not altered by Clifford unitaries, whereas conditional probabilities are reshuffled. This enables the algorithm to effectively target unsampled regions of $G_\mathcal{S}(|\psi\rangle)$. In practice, one can set $U_\mathcal{C}$ as a random Clifford circuit of depth D and evolve the MPS to obtain $|\psi'\rangle$. D should remain small to avoid excessively increasing the bond dimension of $|\psi'\rangle$ [which grows as $\exp(D)$]. Note that once the sampling of $|\psi'\rangle$ is completed one has to map back the sampled Pauli strings in order to find stabilizer strings of the original state, since $G_\mathcal{S}(|\psi\rangle)=U_\mathcal{C}G_\mathcal{S}(|\psi'\rangle)U_\mathcal{C}^\dagger$. This task can be easily achieved at cost $\mathcal{O}(N^2)$ in the tableau formalism, because it involves only applying Clifford unitaries [19,49]. In practice, one iterates the sampling of modified states $|\psi'\rangle$ for several random $U_\mathcal{C}$. At each iteration, one has to collect the newly sampled stabilizer Pauli strings, incorporate them into the tableau containing previously sampled generators, and apply Gaussian elimination to find a new set of independent generators of $G_\mathcal{S}(|\psi\rangle)$. The total number of

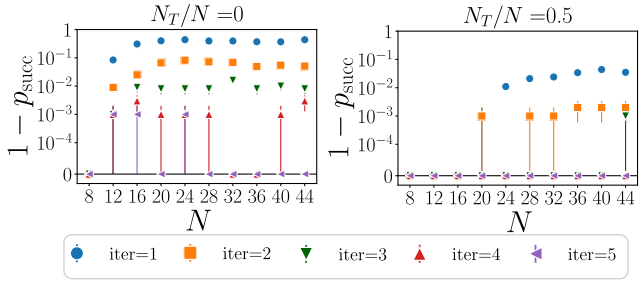


FIG. 1. (One minus) the probability of correctly collecting all $N-N_T$ stabilizer generators of $|\psi\rangle = U_{\mathcal{C}}|N,N_T\rangle$. Different symbols refer to iterations 1,2,3,4,5 over modified states.

these, which we dub k, can only increase with each iteration (converging to the true value k_ψ).

Numerical experiments.—We prepare the state $|N,N_T\rangle \equiv |0\rangle^{\otimes N-N_T}|T\rangle^{\otimes N_T}$, where $|T\rangle = \mathrm{T}(\mathrm{H}|0\rangle) = (|0\rangle + e^{i\pi/4}|1\rangle)/\sqrt{2}$ is a single-qubit magical state. Afterward, we apply a random Clifford circuit $U_{\mathcal{C}} \in \mathcal{C}_N$ obtaining $|\psi\rangle = U_{\mathcal{C}}|N,N_T\rangle$. By construction $k_\psi = N - N_T$, and the stabilizer generators of $|\psi\rangle$ can be obtained by evolving the generators $\{\sigma_1^3 \ldots \sigma_{N-N_T}^3\}$ of $G_{\mathcal{S}}(|N,N_T\rangle)$ within the tableau formalism. $U_{\mathcal{C}}$ has depth N, and each layer consists of gates selected randomly from the Clifford generators {H,S,CNOT}. We contract the circuit to obtain $|\psi\rangle$ as an MPS, and we apply our method to detect its stabilizer generators.

In Fig. 1, we represent (one minus) the probability $p_{\rm succ}$ of correctly obtaining $k = k_\psi$ as N increases, for $\mathcal{N} = 10^3$ and two values of N_T/N (other values, not shown here, have been tested, with similar results). $p_{\rm succ}$ is assessed through the iteration of the method over 10^3 realizations of $U_{\mathcal{C}}$. For each case, we consider 5 iterations over modified states (with $D = 1$). Notice that at the final iteration, for all values of N_T/N, we achieve $p_{\rm succ} \simeq 1$ within the statistical uncertainty, meaning that our technique is always able to learn entirely $G_{\mathcal{S}}(|\psi\rangle)$. In Fig. 2, we represent the number k of generators found by our method as a function of subsequent iterations (iter $= 1, 2, 3, \ldots$) over modified states for various sample sizes $\mathcal{N}$. We set $N = 50$ and $N_T = 25$, so that $k_\psi = N - N_T = 25$, and we examine 10^3 realizations of $U_{\mathcal{C}}$. In this case, MPS bond dimension increases up to $\chi \sim 64$. We observe that even with $\mathcal{N} \sim o(10)$, performing around 10 iterations over modified states is enough to learn the complete stabilizer group.

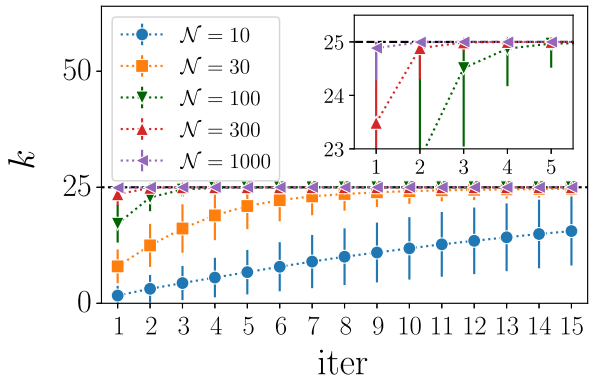


FIG. 2. Number of discovered generators k for successive iterations (iter) of the algorithm over modified states and different sample sizes $\mathcal{N}$ (here, $N = 50$, $N_T = 25$, $k_\psi = 25$). In the inset, a zoom of the same plot is displayed.

Afterward, we examine a doped circuit consisting of random Clifford layers, interleaved with layers containing a constant number τ of T gates placed on random sites. Clifford layers have a staircase geometry, and gates are uniformly sampled from the two-qubit Clifford group [49]. We consider the initial state $|0\rangle^{\otimes N}$ and we investigate how its stabilizer group, which has dimension N, is reduced by the application of T gates. In Fig. 3, we show the number of generators k as a function of the discrete circuit time $n = 0, 1, 2, 3, \ldots$ for $\tau = 3$ and $N = 15$, 30, 45. Data are averaged over many circuit realizations (trajectories). MPS bond dimension increase up to $\chi \sim 1024$ for $N = 45$. Dashed lines represent the minimum possible number of generators at time n, namely, $k_{\rm min}(n) = N - n\tau$ (see Lemma 1 in Supplemental Material [39]). Data show that in typical circuit realizations the value of k at step n fluctuates above this line. This leads to a prolonged preservation of certain stabilizer symmetries for a time n longer than the theoretically required minimum time $n_{\rm min} = N/\tau$. A suitable rescaling of n and k (average) with N and $n_{\rm min}$, respectively (see inset), suggests that this effect might vanish in the thermodynamic limit, whereas fluctuations of k might still be relevant.

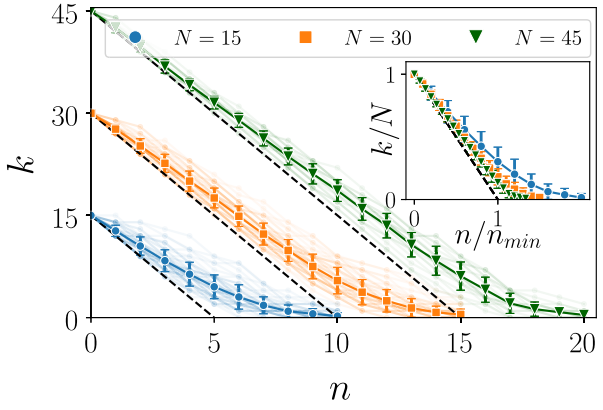


FIG. 3. Number of stabilizer generators k for a state evolving through a doped quantum circuit, which comprises a random Clifford layer followed by a layer of $\tau = 3$ T gates. Integer n is the discrete time of the circuit (i.e., $n = 1$ after applying a random Clifford layer followed by a T layer). Pale colors represent single trajectories ($N_{\rm traj} = 60$ for $N = 15$, 30, $N_{\rm traj} = 20$ for $N = 45$), bold lines are averages. Inset: same plot with k rescaled by N and n rescaled by $n_{\rm min}$.

Conclusions and outlook.—We introduced an effective classical method to learn the stabilizer group of a given MPS. Stabilizer strings are extracted via a biased sampling in the Pauli (Bell) basis over the MPS. During the sampling, on the flight, we discard all Pauli substrings $\boldsymbol{\sigma}_{[1,i]}$ at site i such that $\pi_\rho(\boldsymbol{\sigma}_{[1,i]}) < 1/(2^i\chi_i)$, relying on an exact theoretical argument. Manipulating the MPS has a computational complexity $\mathcal{O}(\chi^3)$, implying no severe constraints on the bond dimension χ. In comparison, our method surpasses approaches based on the MPS exact replica trick, where the scaling would be at least $\mathcal{O}(\chi^6)$ (due to the bond dimension of the two replica MPSs being χ^2). We have shown the effectiveness of our algorithm in T-doped states after information scrambling induced by a Clifford circuit. In addition, we analyzed a prototypical case of chaotic nonequilibrium dynamics induced by the interplay of local magic gates and entangling Clifford layers. For different system sizes and up to bond dimension $\chi \sim 10^3$, we studied the dynamical depletion of the stabilizer group. We have shown that the signature of *stabilizerness*, which is theoretically lower bonded by $N - \tau n$ during the discrete time steps n, definitively survives for longer times with nontrivial fluctuations over the trajectories. Additional investigations may involve comparing our method with the recently introduced Bell difference sampling. This method ensures an overestimation of the stabilizer group with only $\mathrm{pol}(N)$ Bell samples [50]. With our method's ability to systematically access high-weight regions in the distribution $\Pi_\rho(\boldsymbol{\sigma})$, there is potential to extend its application beyond learning just the stabilizer group, but also the state $|\tilde{\psi}\rangle$ such that $|\psi\rangle = U_{\mathcal{C}}|0\rangle^{\otimes k_\psi}|\tilde{\psi}\rangle$, without requiring a full tomography [50]. Our advancements could enable hybrid MPS-stabilizer techniques, providing new methods for classically simulating complex quantum states using knowledge of the stabilizer group.

Finally, let us examine the experimental feasibility of our algorithm: as a matter of fact, the main challenge is the postselection problem due to the need to estimate partial or conditional probabilities; however, a potential solution could involve initially learning the MPS representation of the (low-entangled) state [51], then followed by applying our algorithm on a classical computer.

Note added.—Recently, Ref. [52] appeared, where they made use of a different approach, based on a compressed-MPS folding technique, to reduce the cost of the exact replica trick in computing nonstabilizerness.

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