Giant Graviton Expansion from Bubbling Geometry: Discreteness from Quantized Geometry

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The superconformal index of half-BPS states in $\mathcal{N}=4$ supersymmetric Yang-Mills with gauge group U(N) admits an expansion in terms of giant gravitons, $\mathcal{I}_N(q)=\mathcal{I}_\infty(q)\sum_{m=0}^{\infty}q^{mN}\hat{\mathcal{I}}_m(q)$, where m is the number of giant gravitons and $\mathcal{I}_\infty(q)$ is the graviton index. The expansion can be viewed as the implementation of trace relations for finite N. We derive this expansion directly in supergravity from the class of half-BPS solutions due to Lin, Lunin, and Maldacena in type IIB supergravity. The moduli space of these configurations can be quantized using covariant quantization methods. We show how this quantization leads to the precise expression for the expansion in terms of giant gravitons. Our proposal provides a derivation of the giant graviton expansion directly in terms of quantized supergravity degrees of freedom, and it recovers discrete data via quantum geometries that are classically nonsmooth.

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Introduction.—The AdS/CFT correspondence posits an equivalence between quantum field theories and gravity [1]. In some cases, this correspondence can be viewed as a classical-quantum duality, where classical gravity can be used to explore quantum aspects of the field theory, such as anomalies [2] and entanglement entropy [3]. In its strongest form, both sides are quantum, and thus AdS/CFT provides a framework for a complete theory of quantum gravity.

The prototypical example of AdS/CFT is the duality between four-dimensional $\mathcal{N}=4$ supersymmetric-Yang-Mills theory with U(N) gauge group and type-IIB supergravity on $\mathrm{AdS}_5\times S^5$. In the large-N limit, the bulk is well described by classical gravity. However, as pointed out early on, understanding finite-N corrections is crucial to ultimately elucidating discreteness of the spectrum in the gravitational theory, and is central to answering black hole puzzles such as the information paradox [4].

Discreteness naturally shows up in the spectrum of states of a compact theory, such as on $S^1\times S^3$. Powerful results for enumerating states of $\mathcal{N}=4$ SYM can be obtained within the sector of supersymmetric, i.e., BPS, states. Such states are counted by the superconformal index, which is a refined Witten index

$$\mathcal{I}_N(q_i)=\mathrm{Tr}\left[(-1)^F e^{-\beta\mathcal{H}}q_i^{J_i}\right]. \tag{1}$$

Here $(-1)^F$ is the Fermion number operator and $\mathcal{H}=\frac{1}{2}\{\mathcal{Q},\mathcal{Q}^\dagger\}$, where $\mathcal{Q}$ is one of the supercharges. The refinement comes from introducing fugacities q_i with corresponding charges J_i commuting with $\mathcal{Q}$. Because of the supersymmetric cancellation between bosons and fermions, only BPS states with $\mathcal{H}=0$ contribute to the index.

In this Letter, we reproduce the $\frac{1}{2}$-BPS version of this index from the dual supergravity point of view. Since the index counts states in the field theory, the natural framework here is to count the corresponding $\frac{1}{2}$-BPS configurations in IIB supergravity. These solutions were obtained by Lin, Lunin, and Maldacena (LLM) in Ref. [5], and are commonly referred to as bubbling solutions. These LLM solutions are smooth geometries (assuming appropriate boundary conditions) and can be thought of as D3-branes dissolved into fluxes much as $\mathrm{AdS}_5\times S^5$ is obtained in the near horizon limit of a stack of D3-branes. We develop a complete understanding of the finite-N index directly in supergravity and provide insights into gravitational state counting that can be further applied to the study of black hole microstates and quantum gravity.

The giant graviton expansion.—At leading order, AdS/CFT pertains to the large-N limit of the field theory. In this limit, the field-theoretic index reduces to a simple form: $\mathcal{I}_N(q)\to\mathcal{I}_\infty(q)$, where we have focused on a single fugacity, q. Finite-N corrections arise from trace relations removing states from the spectrum. An explicit calculation of the $\frac{1}{2}$-BPS index suggested a finite-N expansion of the form [6,7]

$$\mathcal{I}_N(q)=\mathcal{I}_\infty(q)\left(1+\sum_{m=1}^{\infty}q^{mN}\hat{\mathcal{I}}_m(q)\right). \tag{2}$$

This structure of finite-N corrections is known as the giant graviton expansion of the index, and much recent work has been devoted to understanding its features on both sides of the AdS/CFT correspondence [8–16].

On the gravitational side, $\mathcal{I}_\infty(q)$ has a clear interpretation as the multigraviton index, counting spin-2 fluctuations in gravity [17]. This component of the index is independent of N. The finite-N dependence shows up in the factors multiplying $\hat{\mathcal{I}}_m(q)$, which is interpreted as the index for the world volume theory of a stack of m D3-branes wrapped on an S^3 inside S^5 [8–10]. The wrapped D3-branes are stabilized by angular momentum on S^5, and were called giant gravitons in [18]. In the following, we demonstrate how an accounting of gravitational solutions can precisely reproduce both $\mathcal{I}_\infty(q)$ and $\hat{\mathcal{I}}_m(q)$.

The $\frac{1}{2}$-BPS index.—The $\frac{1}{2}$-BPS states have $\Delta = J$ where Δ is the conformal dimension, and J is one of the Cartan generators of the SU(4) R symmetry. Counting such states leads to

$$\mathcal{I}_N(q) = \mathrm{Tr}[(-1)^F q^J] = \prod_{n=1}^{N} \frac{1}{1-q^n} = \frac{1}{(q)_N}, \qquad (3)$$

where $(q)_m = \prod_{j=1}^{m}(1-q^j)$ is the Pochhammer symbol. As a consequence of the q-binomial theorem, this may be expanded as [19]

$$\mathcal{I}_N(q) = \mathcal{I}_\infty(q) \sum_{m=0}^{\infty} (-1)^m \frac{q^{\frac{m(m+1)}{2}}}{(q)_m} q^{mN}, \qquad (4)$$

which admits an interpretation in terms of giant gravitons due to the term q^{mN} [8], where N is viewed as the tension of one giant graviton.

By manipulating the Pochhammer symbol, the giant graviton expansion can be brought into the suggestive form

$$\mathcal{I}_N(q) = \mathcal{I}_\infty(q)\left(1 + \sum_{m=1}^{\infty} q^{mN} \frac{1}{(q^{-1})_m}\right). \qquad (5)$$

Comparison with (2) allows us to identify the m giant graviton index as

$$\hat{\mathcal{I}}_m(q) = \frac{1}{(q^{-1})_m} = \mathcal{I}_m(q^{-1}). \qquad (6)$$

The authors of [16,20] provided further insight into the giant graviton expansion by showing that this expansion indeed arises from considering the probe limit of giant gravitons as D3-branes and semiclassical quantization around the probe solution. Here, we instead recover the giant graviton expansion of Eq. (5) directly from a fully backreacted bubbling geometry.

The IIB geometry of giant gravitons.—In seeking a holographic understanding of the $\frac{1}{2}$-BPS giant graviton expansion, (5), one is led to consider the $\frac{1}{2}$-BPS configurations in supergravity known as the LLM bubbling solutions of type IIB supergravity with only the metric and self-dual five-form active [5]. Because of the preserved $\mathrm{SO}(4) \times \mathrm{SO}(4) \times \mathbb{R}$ isometry, these solutions depend on three coordinates, x_1, x_2, and y. Remarkably, the complete solution is determined in terms of a single harmonic function, $z(x_1, x_2, y)$, which obeys the equation

$$\partial_i \partial_i z + y \partial_y \left(\frac{\partial_y z}{y}\right) = 0. \qquad (7)$$

LLM demonstrated that the solution is non-singular so long as $z = \pm\frac{1}{2}$ on the $y = 0$ plane. With these boundary conditions at $y = 0$, the solution to the Laplacian, (7), is unique, and the IIB solution is fully determined.

The essential point is that LLM geometries are specified by two colorings [21] of the (x_1, x_2) plane, which we will refer to as droplets. Let $\mathcal{D}$ denote the region of the $y = 0$ plane, where $z = -\frac{1}{2}$ with boundary $\partial\mathcal{D}$. The complement $\mathcal{D}^c$ has $z = +\frac{1}{2}$. If the droplets are of finite size, then the spacetime is asymptotically $\mathrm{AdS}_5 \times S^5$. In particular, $\mathrm{AdS}_5 \times S^5$ corresponds to a disk of radius R in $\mathcal{D}$, where $R = L^2$ with L being the AdS radius. Giant gravitons correspond to a disk with some droplets missing, and dual giants correspond to droplets outside the disk [5]. Maximal giants, i.e., those with maximum angular momentum, correspond to a droplet in the center of the AdS disk.

While the LLM geometries are classical solutions to IIB supergravity, quantization of the self-dual five-form flux leads to a quantization of the area of $\mathcal{D}$ in integer units of $(2\pi)^2 \ell_p^4$, where ℓ_p is the Planck length. Introducing $\hbar = 2\pi \ell_p^4$, flux quantization becomes

$$N = \frac{1}{2\pi\hbar} \int_{\mathcal{D}} \mathrm{d}^2 x \in \mathbb{N}. \qquad (8)$$

Here, N is identified with the flux of $F_{(5)}$ through S^5, or, equivalently, the number of D3-branes that have dissolved into fluxes. This choice of an effective $\hbar$ is motivated by thinking of the (x_1,x_2) plane as phase space with minimum area $2\pi\hbar$. Flux quantization also requires that each $z = +\frac{1}{2}$ droplet inside of $\mathcal{D}$ be quantized $m_i \in \mathbb{N}$, where m_i is interpreted as the number of giant gravitons at position i with $m = \sum_i m_i$ the total number of giant gravitons.

The energy Δ and angular momentum J may be extracted from the asymptotic behavior of the metric [5]:

$$\Delta = J = \frac{1}{4\pi\hbar^2}\left[\int_{\mathcal{D}} \mathrm{d}^2 x (x_1^2 + x_2^2) - \frac{1}{2\pi}\left(\int_{\mathcal{D}} \mathrm{d}^2 x\right)^2\right]. \qquad (9)$$

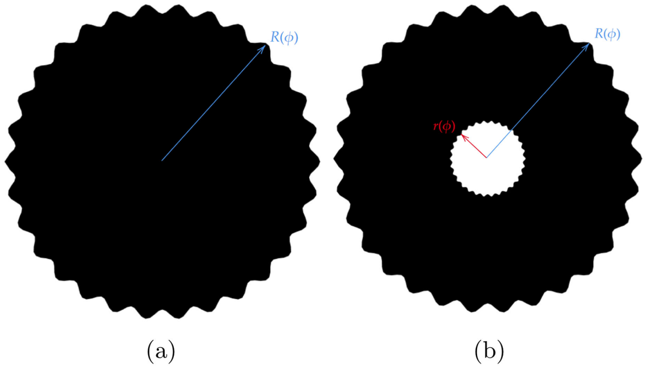

FIG. 1. The LLM description of $\mathrm{AdS}_5 \times S^5$ is given by a disk of radius R. Here we schematically portray fluctuations about this background. (a) Graviton fluctuations parametrized by the curve $R(\phi)$. (b) Maximal giant with both graviton fluctuations on the outer boundary parametrized by $R(\phi)$ and fluctuations of the maximal giant on the inner boundary parametrized by $r(\phi)$.

The angular momentum J is the dual of the R charge in field theory, and we denote the corresponding fugacity as q.

Quantization of LLM moduli space and the giant graviton expansion.—To holographically reproduce the $\frac{1}{2}$-BPS index, we would like to count supergravity states at fixed N. This corresponds to holding the quantized area of the region $\mathcal{D}$ fixed according to (8) while allowing both fluctuations of the boundary, $\partial\mathcal{D}$, and topology change. As observed in [5], boundary fluctuations, as shown in Fig. 1(a), correspond to graviton modes, while giant gravitons change the topology of the solution. Maximal giants with fluctuations are depicted in Fig. 1(b). Classically, these fluctuations live in a continuous moduli space. However, they were quantized in [22] using the covariant quantization method of [23,24].

To quantize giant graviton topologies, we consider the case of covariant quantization with disjoint boundary, $\partial\mathcal{D}$. We are interested in the case that $\partial\mathcal{D}$ has a set of collected components labeled by B such that $\partial\mathcal{D} = \bigcup_{b\in B}\partial\mathcal{D}^{(b)}$. Assume that $\partial\mathcal{D}^{(b)}$ is described by a closed curved $\gamma^{(b)}(s)$ and let $\delta\gamma_\perp^{(b)}(s)$ denote the outward-directed variation of $\partial\mathcal{D}^{(b)}$ in the normal direction at a point $s\in\partial\mathcal{D}^{(b)}$. When $\gamma^{(b)}(s)$ is described by a single-valued curve $r(\phi)$, these are related by

$$\frac{\mathrm{d}s}{r(\phi)\mathrm{d}\phi} = \frac{\delta r}{\delta\gamma_\perp^{(b)}}. \tag{10}$$

This then has a symplectic form [22] and correspondingly satisfies a Poisson bracket

$$\left\{\delta\gamma_\perp^{(b)}(s), \delta\gamma_\perp^{(\tilde{b})}(\tilde{s})\right\} = 2\pi\hbar\delta'(s-\tilde{s})\delta_{b\tilde{b}}. \tag{11}$$

Importantly, different droplet boundaries are completely decoupled. This is subject to the constraint of droplet area quantization

$$\oint_{\gamma^{(b)}} \mathrm{d}s\,\delta\gamma_\perp^{(b)}(s) = 0, \tag{12}$$

which specifies symplectic sheets in the moduli space of droplets.

We restrict our focus to maximal giants only. These maximal giants are all overlapping and centered at the origin of the LLM plane and hence yield a configuration of the form shown in Fig. 1(b), with the area of the central hole related to the number of maximal giants. For a configuration of m maximal giant gravitons, flux quantization requires

$$N = \frac{R^2 - r^2}{2\hbar}, \qquad m = \frac{r^2}{2\hbar}. \tag{13}$$

We may parametrize the outer boundary by a curve $R(\phi)$ and the inner boundary by a curve $r(\phi)$, as shown in Fig. 1(b), such that

$$\begin{aligned} R(\phi)^2 &= \sum_{n\in\mathbb{Z}}\alpha_n e^{in\phi}, \qquad \alpha_0 = R^2, \qquad \alpha_{-n} = \alpha_n^*, \\ r(\phi)^2 &= \sum_{j\in\mathbb{Z}}\beta_j e^{ij\phi}, \qquad \beta_0 = r^2, \qquad \beta_{-j} = \beta_j^*. \end{aligned} \tag{14}$$

This parametrization automatically satisfies the area-preserving constraint, (12). Moreover, after promoting Poisson brackets to commutators, (11) requires that the modes satisfy

$$[a_m, a_n] = m\delta_{m+n}, \qquad [b_m, b_n] = m\delta_{m+n}, \tag{15}$$

where $a_m = \alpha_m/(2\hbar)$ and $b_j = \beta_j/(2\hbar)$ are the normalized operators. Using (13), the maximal giants then have charges

$$\Delta = J = mN + \sum_{n\geq 1}a_{-n}a_n - \sum_{j\geq 1}b_{-j}b_j. \tag{16}$$

As a result, the index computed from the gravitational data takes the form

$$\mathrm{Tr}q^J = \sum_{m=0}^{\infty} q^{mN}\left(\prod_{n\geq 1}\sum_{N_n\geq 0} q^{nN_n}\right)\left(\prod_{j\geq 1}\sum_{N_j\geq 0} q^{-jN_j}\right). \tag{17}$$

The first term on the right corresponds to $J = mN$, which is the "classical" angular momentum of m maximal giants. The second term originates from the a_n fluctuations, with N_n denoting the occupation number for each mode. This term corresponds to quantized multigraviton fluctuations on the outer boundary of the LLM disk, giving $\mathcal{I}_\infty(q) = 1/(q)_\infty$ as expected.

Now consider the third term, which is the contribution of the giant graviton fluctuations, $\hat{\mathcal{I}}_m(q)$. We may not make the approximation that the inner boundary is large enough to support arbitrary b_j fluctuations, as the size of the inner boundary is associated with $\hbar$. Indeed, observe that b_j changes J by j, and so increases r^2 by $2\hbar j$. Consequently, the inner radius is

$$r(\phi)^2 = 2\hbar\left(m + \sum_j N_j e^{ij\phi}\right), \tag{18}$$

where N_j is the b_j occupation number. If $\sum_j N_j > m$, then the radius squared can go negative. To avoid this, we must cut off the sum of occupation numbers at m. The coefficient of each term q^{-n} in the expansion of $\hat{\mathcal{I}}_m(q)$ should therefore count the sets of occupation numbers satisfying

$$\sum_{j=0}^{\infty} N_j j = n, \tag{19}$$

with the additional constraint on the total sum, $\sum_j N_j \leq m$. This counting is precisely the number of partitions of n into at most m parts. By a standard result from the theory of partitions, the same result is obtained by counting the number of partitions of n with no part greater than m. Therefore,

$$\hat{\mathcal{I}}_m(q) = \prod_{j=1}^{m} \frac{1}{1-q^{-j}} = \frac{1}{(q^{-1})_m} = \mathcal{I}_m(q^{-1}), \tag{20}$$

giving the full index

$$\mathcal{I}_N(q) = \mathcal{I}_\infty(q) \sum_{m=0}^{\infty} \mathcal{I}_m(q^{-1}) q^{mN}. \tag{21}$$

In particular, we see that (21), derived from quantizing the LLM description, matches the giant graviton expansion (5).

The Fermi droplet picture and the boundary Hamiltonian.—Although we have focused on semiclassical quantization of bubbling geometries, one may take a complementary approach to $\frac{1}{2}$-BPS states. LLM solutions are dual to a subsector of chiral primaries in $\mathcal{N} = 4$ super Yang-Mills with $\Delta = J$ [5]. These admit a description in terms of free fermions in a harmonic oscillator potential [25,26]. In particular, the "droplets" in the (x_1, x_2) plane in the supergravity description precisely correspond with droplets in the free fermion phase space. We now consider the effective picture of quantizing this fermion liquid.

Consider a droplet of m (not necessarily maximal) giant gravitons. Then this will have charges

$$\Delta = J = \frac{r^2}{4\hbar^2}(R^2 - r^2 - \xi^2), \tag{22}$$

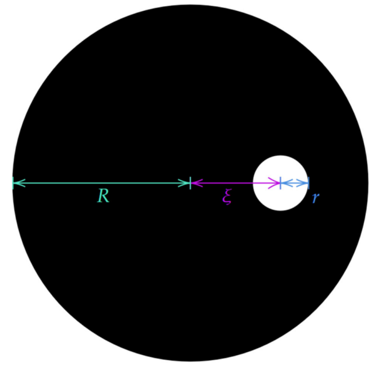

FIG. 2. Schematic depiction of giants in the LLM droplet picture. When $\xi = 0$, these become maximal giants.

where ξ denotes the distance from the origin of the AdS disk, R denotes the radius of the AdS disk, and r denotes the radius of the giant(s) (see Fig. 2). Because of flux quantization, this may be recast as

$$\Delta = J = m(N - p) = mp', \tag{23}$$

where $p = \xi^2/2\hbar$ is the quantization of ξ^2, and $p' = 1, \ldots, N$ is a convenient choice of angular momentum quantization [27]. Then, we have a counting problem of picking occupation numbers n_p for the N angular momentum levels. To get the contribution of m giant gravitons, we impose the following constraint:

$$n_1 + \cdots + n_N = m, \tag{24}$$

and then sum over m. It can be seen that this is the q analog of the classic balls and bins problem. Thus, we must put m giant gravitons into N angular momentum levels, whose solution is

$$\mathcal{I}_N(q) = \sum_{m=0}^{\infty} \begin{bmatrix} N+m-1 \\ m \end{bmatrix}_q q^m, \tag{25}$$

where the brackets denote the q-binomial coefficients. We may observe that (25) can be rewritten as

$$\mathcal{I}_N(q) = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{q^{m-j}}{(q)_{m-j}} \frac{(-1)^j q^{\frac{j(j+1)}{2}}}{(q)_j} q^{jN}, \tag{26}$$

which follows from the q-binomial theorem. Applying the discrete version of Fubini's theorem, we may swap the order of the sums. One can do the sum over m using the q expansion

$$\mathcal{I}_\infty(q) = \frac{1}{(q)_\infty} = \sum_{k=0}^{\infty} \frac{q^k}{(q)_k}, \tag{27}$$

to get that

$$\mathcal{I}_N(q) = \mathcal{I}_\infty(q) \sum_{j=0}^{\infty} \frac{(-1)^j q^{\frac{j(j+1)}{2}}}{(q)_j} q^{jN}, \tag{28}$$

which is precisely the expansion (4). However, j is not directly interpreted as the number of giant gravitons (although it is related).

The Fermi droplet picture leads to further geometric insight. The authors of [28] used deformation quantization to obtain a Hamiltonian

$$H = \sum_{m=0}^{\infty} c_m \left(m + \frac{1}{2}\right) - \frac{N^2}{2}, \quad \sum_{m=0}^{\infty} c_m = N, \quad c_m \in \{0,1\}. \tag{29}$$

This is equivalent to the Hamiltonian of N free fermions in a harmonic oscillator potential as follows:

$$H = \sum_{i=1}^{N} \left(f_i + \frac{1}{2}\right) - \frac{N^2}{2}, \quad 0 \leq f_1 < f_2 < \cdots < f_N < \infty, \tag{30}$$

which may then be mapped onto our Fermi droplet picture by identifying [29]

$$n_N = f_1, \quad n_{N-i} = f_{i+1} - f_i - 1, \quad i = 1, 2, \ldots, N-1, \tag{31}$$

which precisely reproduces our earlier Hamiltonian $H = \sum_{p=1}^{N} p n_p$. The counting (24) then corresponds to fixing $f_N = N + m - 1$.

Let us now discuss the central role of nonsmooth quantized geometries as follows from these Hamiltonians. The coloring of the (x_1, x_2) plane in [28] is given by

$$z(x_1, x_2) = \frac{1}{2} - \sum_{n=0}^{\infty} c_n \phi_n(x_1, x_2),$$
$$\phi_n(x_1, x_2) = 2(-1)^n e^{-r^2/\hbar} L_n\left(\frac{2r^2}{\hbar}\right). \tag{32}$$

Even though ϕ_n is azimuthally symmetric, these solutions generate the most general one via $\mathrm{U}(\infty)$ transformations as discussed in [30] and we proceed to explore their implications for the gravitational picture. States of the form

$$c_{n<n_0} = 0, \qquad c_{n_0 \leq n \leq n_1} = 1, \qquad c_{n>n_1} = 0, \tag{33}$$

correspond to (nonfluctuating) maximal giant gravitons. The rest correspond to "new geometries" in the LLM language.

As shown in Fig. 3, there is an inherent fuzziness of the colorings in (32). This reflects the fact that the

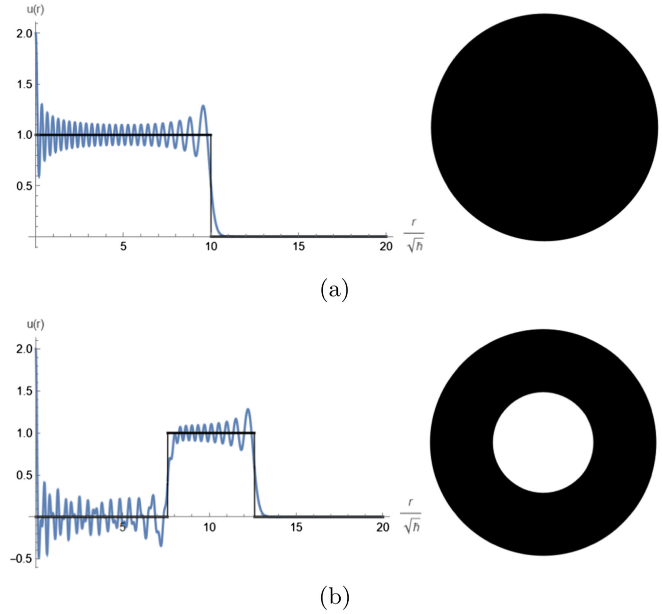

FIG. 3. Plots of $u = \frac{1}{2} - z$ as a function of $r/\sqrt{\hbar}$ in blue and their corresponding approximate classical interpretation in the LLM moduli space in black. We have set $N = 50$ for visualization. (a) corresponds to $c_1 = \cdots = c_{50} = 1$ with all other c_n vanishing, and is identified with $\mathrm{AdS}_5 \times S^5$. (b) corresponds to $c_{31} = \cdots = c_{80} = 1$ and all other c_n vanishing, and can be identified with a maximal giant.

classical geometry breaks down when quantizing, and we are left with a quantum geometry. A precedent in this direction appeared in the context of superstar geometries in [31,32].

Conclusions.—In order to obtain a discrete spectrum, the gravitational theory must necessarily be quantized. Since we have focused on the $\frac{1}{2}$-BPS states, this originates from D3 branes in $\mathrm{AdS}_5 \times S^5$. There are various complementary approaches to the quantization of such states. The index can be obtained in the D3-brane world volume theory [16,20], by using the Fermi droplet picture, or by quantizing smooth LLM geometries, as we have shown here. Quantizing the geometry leads to insight into the importance of nonsmooth configurations.

It would be interesting to extend the enumeration of gravity states to indices with less supersymmetry, such as the $(1/16)$-BPS index. The goal is to understand the $(1/16)$-BPS black holes (with singularities protected by horizons) [33,34] in terms of $(1/16)$-BPS bubbling geometries in the same way D3-branes dissolve into LLM geometries. One could then extend the present analysis to the bubbling black hole case, and in this manner take us one step closer to a complete understanding of black hole microstates and the nature of quantum gravity.

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