New Classical Integrable Systems from Generalized $T\bar{T}$-Deformations

Benjamin Doyon,[1] Friedrich Hübner,[1] and Takato Yoshimura[2,3]
[1]*Department of Mathematics, King's College London, Strand, London WC2R 2LS, United Kingdom*
[2]*All Souls College, Oxford OX1 4AL, United Kingdom*
[3]*Rudolf Peierls Centre for Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, United Kingdom*

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We introduce and study a novel class of classical integrable many-body systems obtained by generalized $T\bar{T}$ deformations of free particles. Deformation terms are bilinears in densities and currents for the continuum of charges counting asymptotic particles of different momenta. In these models, which we dub "semiclassical Bethe systems" for their link with the dynamics of Bethe ansatz wave packets, many-body scattering processes are factorized, and two-body scattering shifts can be set to an almost arbitrary function of momenta. The dynamics is local but inherently different from that of known classical integrable systems. At short scales, the geometry of the deformation is dynamically resolved: either particles are slowed down (more space available), or accelerated via a novel classical particle-pair creation and annihilation process (less space available). The thermodynamics both at finite and infinite volumes is described by the equations of (or akin to) the thermodynamic Bethe ansatz, and at large scales generalized hydrodynamics emerge.

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Introduction.—Since the inception of generalized hydrodynamics (GHD) in 2016 [1,2], there has been a resurgence of interest in understanding the nature of the dynamics in integrable systems. GHD has proven to be a powerful and universal tool that captures it at large scales, and its predictions have been tested against cold-atom experiments in different platforms [3–5]. GHD has also been studied beyond the quantum realm and applied to classical systems including various types of hard rods [6–8], the Toda chain [9,10], the nonlinear Schrödinger equation [11,12], the Calogero-Moser model [13] and the sinh- and sine-Gordon models [14–16]. It provides the statistical framework [17] for the theory of soliton gases [18,19]. The structure of GHD is extremely general, and it requires only limited data from the underlying system, such as the two-body scattering shift.

Yet, a full understanding of how GHD emerges from the microscopic dynamics is still lacking. In the hard-rod and box-ball systems, rigorous proofs from slowly varying ensembles are available [8,20,21]: in soliton gases they are obtained from finite-gap solutions [18,22–24], *ab initio* derivations exist from kinetic theory [25] and Bethe-ansatz semiclassical principles [26], and the equations of state are well understood [27,28]. But every model has specific properties. To understand the universality of GHD, it is important to construct *new integrable systems that can access the full space of GHD equations*.

In this Letter, we do just that. We define a new class of classical many-particle systems with short-range interactions that are integrable, and that cover a *very large space of scattering functions*. The new systems are shown to arise from *generalized $T\bar{T}$ deformations*. $T\bar{T}$ deformations were introduced in relativistic quantum field theory [29–31] as integrability-preserving deformations based on local conserved currents that modify the scattering matrix by "Castillejo-Dalitz-Dyson factors" [30,31]. Matrix elements of local fields have been recently studied [32–34], and $T\bar{T}$ deformations have been adapted to systems of different kinds [35–41]; see the review [42]. Here, "generalized $T\bar{T}$ deformations" are those proposed in [43], based on the larger space of *extensive* conserved charges first studied in the context of the nonequilibrium dynamics of integrable systems [44–46]. One admits conserved quantities measuring the density of asymptotic momenta, and generalized $T\bar{T}$ deformations modify the scattering matrix by an *arbitrary momentum function*, although no explicit construction was made. Our models provide the explicit construction for classical Galilean particle systems. We confirm that they are Liouville integrable, that many-particle scattering is elastic and factorizes into two-body shifts, and that the two-body shift can be chosen as an almost arbitrary function of momenta.

Constructing effective, fully integrable many-particle Hamiltonians for other objects such as solitons in nonlinear media is an old problem; see, e.g., [47]. Our models are the first to do that, and give in particular the potential for precise initial state preparation in inhomogeneous soliton gases [18,19], a problem of current relevance.

In [39] it was shown that "mass-momentum" $T\bar{T}$ deformations simply change the length of particles. This can also be interpreted as a local change of the *effective space* particles freely travel through, a special case of a geometric interpretation [48] much like in GHD [49], but the exact local properties of (generalized) $T\bar{T}$ deformation are nebulous. We obtain an explicit microscopic dynamics implementing the geometry, generalizing the change of particle lengths. Particles are tracers for their asymptotic momenta and "go through" each other: additional available space is implemented by a slowing down at particles' proximity, while reduced space is accounted for by a novel process of creation and annihilation of pairs of particles and anti-particles, which effectively gives an acceleration.

We further provide an expression for the free energy, showing at infinite volume the thermodynamic Bethe ansatz (TBA) with Boltzmann-Maxwell statistics (see, e.g., [50]); and, remarkably, a similar structure *at finite volumes* generalizing the recent result [51] in hard rods [52] (we are not aware of any other examples). We then show that the GHD equation emerges in the large space-time limit in the generality of arbitrary two-body shift. We confirm this by numerical simulations.

The models we introduce are different from most known classical integrable systems, whose dynamics are not of tracer type. They widely generalize hard-rod gases [6,8], and are closely related to the quantum Bethe ansatz and the gas of Bethe wave packets introduced recently [26]. We refer to them as "semiclassical Bethe systems." Some of the results presented here are proved rigorously in the separate paper [53].

The model.—Consider the N-particle classical phase space with canonical coordinates $(\boldsymbol{y},\boldsymbol{\theta})\in\mathbb{R}^{2N}$, $\{y_i,\theta_j\}=\delta_{ij}$—which will be identified with asymptotic coordinates—and the free-particle Hamiltonian $H(\boldsymbol{y},\boldsymbol{\theta})=\sum_{i=1}^N\theta_i^2/2$. Let $\psi(x,\theta)$ satisfy $\psi(-x,-\theta)=\psi(x,\theta)$ and $|x|\partial_x\psi(x,\theta)\to 0$ ($|x|\to\infty$). The generating function

$$\Phi^{[\psi]}(\boldsymbol{x},\boldsymbol{\theta})=\sum_i x_i\theta_i+\frac{1}{2}\sum_{i,j}\psi(x_i-x_j,\theta_i-\theta_j) \quad (1)$$

induces the following canonical transformation to the "real" coordinates $(\boldsymbol{x},\boldsymbol{p})$ as $\boldsymbol{y}=\nabla_{\boldsymbol{\theta}}\Phi^{[\psi]}$, $\boldsymbol{p}=\nabla_{\boldsymbol{x}}\Phi^{[\psi]}$:

$$y_i=x_i+\sum_{j\neq i}\partial_\theta\psi(x_i-x_j,\theta_i-\theta_j) \quad (2)$$

$$p_i=\theta_i+\sum_{j\neq i}\partial_x\psi(x_i-x_j,\theta_i-\theta_j), \quad (3)$$

where ∂_x (∂_θ) means derivative with respect to the first (second) argument of $\psi(x,\theta)$. The Hamiltonian takes the form

$$H^{[\psi]}(\boldsymbol{x},\boldsymbol{p})=\sum_{i=1}^N\frac{\theta_i(\boldsymbol{x},\boldsymbol{p})^2}{2}=\sum_{i=1}^N\frac{p_i^2}{2}+V^{[\psi]}(\boldsymbol{x},\boldsymbol{p}), \quad (4)$$

where $\theta_i(\boldsymbol{x},\boldsymbol{p})$ are obtained by solving (3) (see below) and the "quasipotential" $V^{[\psi]}(\boldsymbol{x},\boldsymbol{p})$ is defined by the second equation. The trajectories in phase space $t\mapsto[\boldsymbol{x}(t),\boldsymbol{p}(t)]$ are induced from the free dynamics $\boldsymbol{y}\to\boldsymbol{y}(t)=\boldsymbol{y}+\boldsymbol{\theta}t$ via the change of coordinates Eqs. (2) and (3). Note that there could be multiple trajectories that are admissible when $\partial_x\partial_\theta\psi(x,\theta)$ is negative.

Three statements hold: (i) the Hamiltonian (4) is Liouville integrable, i.e., there are N-independent conserved quantities, including the Hamiltonian, that Poisson commute with each other and are nice enough functions of phase space. Indeed a natural set is $Q_a=\sum_i\theta_i(\boldsymbol{x},\boldsymbol{p})^a=\sum_i p_i^a+V_a^{[\psi]}$, $a=1,\ldots N$. (ii) The quasipotentials are short-range. This is in the weak sense that whenever particles lie on well-separated intervals, $A_1,A_2\subset\mathbb{R}$, $\mathrm{dist}(A_1,A_2)\to\infty$, $I_1\cup I_2=\{1,\ldots,N\}$, $\boldsymbol{x}_{I_i}\subset A_i$, then these do not interact, $V_a^{[\psi]}(\boldsymbol{x},\boldsymbol{p})\to\sum_i V_a^{[\psi]}(\boldsymbol{x}_{I_i},\boldsymbol{p}_{I_i})$. An important consequence is that one can define conserved densities $q_a(x)$ such that $Q_a=\int dx\,q_a(x)$, and with $\{q_a(x_1),q_{a'}(x_2)\}\to 0$ for $x_i\in A_i$. Thus, in the limit of N large for physically sensible finite-density distributions, conserved densities commute at large distances. (iii) The multiparticle scattering processes are *elastic*—the sets of incoming and outgoing momenta are the same—and *factorize* into two-body scattering processes. The two-body scattering shift for incoming momenta θ_1, θ_2 is given by $\varphi(\theta_1-\theta_2)$, where $\varphi(\theta)=\lim_{x\to\infty}[\partial_\theta\psi(x,\theta)-\partial_\theta\psi(-x,\theta)]$. Factorized scattering means that in an N-body scattering event, outgoing particle θ is shifted, with respect to the straight trajectory of incoming particle θ, by the sum $\omega(\theta)=\sum_{\theta'\neq\theta}\mathrm{sgn}(\theta'-\theta)\varphi(\theta-\theta')$ of the two-body shifts with all particles that it has crossed. Factorized scattering is a fundamental property of many-body integrable systems [54]. Here, because $\psi(x,\theta)$ becomes constant in x as $x\to\pm\infty$, at long times $p_i\sim\theta_i$, and $x_i\sim y_i(t)+s_i^\pm$ with $s_i^+-s_i^-=\omega(\theta_i)$. Thus, θ_i are asymptotic momenta and y_i are simply related to the impact parameters of the scattering process. Note that each particle i has the same incoming and outgoing momentum θ_i: thus, it is a tracer for where the asymptotic momentum lies at finite times. We call this a "tracer dynamics".

For any real symmetric $\varphi(\theta)$, we may choose $\psi(x,\theta)=f(x)\phi(\theta)$, where $\phi(\theta)=\int_0^\theta d\theta'\,\varphi(\theta')$ and $f(x)$ interpolates between $-1/2$ and $1/2$, e.g., $f(x)=(x/2\sqrt{x^2+\alpha^2})$ for some $\alpha\neq 0$. Therefore, this is a new family of classical Liouville integrable, short-range, factorized-scattering models, covering *a large class of two-body shift functions*. Only a few shift functions are known to date to correspond to classical integrable models, hence this is a large extension. Note that fixing the scattering does not fix the dynamics: there is still freedom in $\psi(x,\theta)$. In [53], assuming that ψ is twice continuously differentiable, and that $\partial_\theta\partial_x\psi(x,\theta) \geq 0$ [thus $\varphi(\theta) \geq 0$] along with a finite-range condition, we rigorously show the statements above, and we show that (2), (3) have unique continuously differentiable solutions. We believe much of this stays true under weaker conditions, but uniqueness can be broken, leading to important physical effects that we discuss below.

By a judicious choice of $\psi(x,\theta)$ one can reproduce the hard-rod dynamics [6,8,39]; see the Supplemental Material [55]. The generating function (1) has the structure of a phase Φ for Bethe wave functions $\Psi = e^{i\Phi}$, with Bethe roots θ_i and dynamics from semiclassical arguments: $\boldsymbol{p} = \nabla_{\boldsymbol{x}}\Phi$ are physical momenta, and $\boldsymbol{y} = \nabla_{\boldsymbol{\theta}}\Phi$ evolve trivially. In the Lieb-Liniger model, gases of wave packets are indeed described by $\psi_{\rm ll}(x,\theta) = \frac{1}{2}\,\mathrm{sgn}(x)\phi(\theta)$ with $\phi(\theta)$ the quantum scattering phase [26], and we expect a similar relation for most quantum many-body integrable systems.

Microscopic dynamics.—The effect of the interaction can be seen as a particle-dependent, dynamical change of metric from x to y space where it is free: the change of infinitesimal length is $dy_i = K_i(x_i)dx_i$ where $K_i(x) = (1 + \sum_{j\neq i}\partial_\theta\partial_x\psi(x - x_j, \theta_i - \theta_j))$ measures the effective "free" space. It can best be pictured in the two-particle case; see Fig. 1 and [55]. For $\varphi(\theta) > 0$ (e.g., $\psi_{\rm ll}$) particles slow

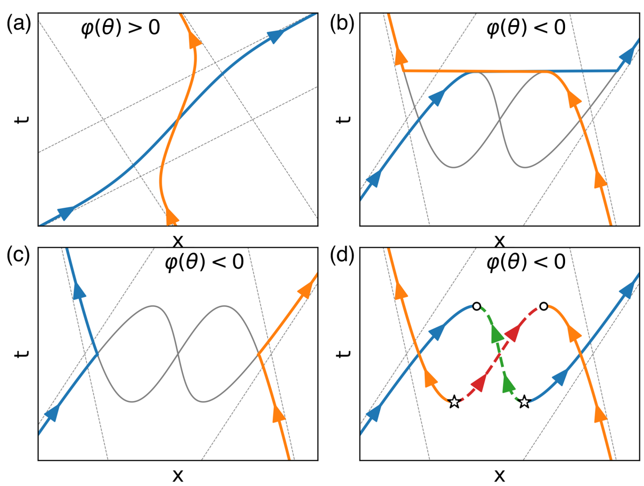

FIG. 1. Trajectories of two particles during scattering for (a) $\varphi(\theta) > 0$ and (b)–(d) $\varphi(\theta) < 0$. The thin dashed lines are the asymptotic trajectories. We show three interpretations of the multivalued solutions for $\varphi(\theta) < 0$; (b) particles jump instantaneously; (c) particles are relabeled during collisions; (d) spontaneous creation of a particle-antiparticle pairs: starting from the bottom, at a certain time (marked by a star) the blue and orange particles are close enough to allow for two pair creations [there are three solutions: two blue-orange (outer), one red-green (inner)]. Later one particle of each pair (the red and green) each annihilates one of the original particles (marked by a circle), leaving the new blue and orange particles as outgoing.. We used $\psi(x,\theta) = (x/\sqrt{x^2+\alpha^2})2\arctan(\theta/c)$, with $c = 0.5$, $\alpha = 0.4$ for (a) and $c = -1$, $\alpha = 1$ for (b)–(d).

down during scattering, giving an effective backward displacement [Fig. 1(a)] interpreted as the presence of additional, hidden space where particles must travel. If $\psi(x,\theta) = \frac{1}{2}\,\mathrm{sgn}(x)\phi(\theta)$ for some $\phi(\theta)$, they "stick" and acquire an internal clock that accounts for this extra space at collisions [26]. For $\varphi(\theta) < 0$, (2) does not necessarily have a unique solution. In case it does, particles speed up, giving an effective forward displacement and a reduction of effective space; hard rods of positive lengths are a limiting case, where the displacement—which traces the momentum being transferred—is instantaneous. But for most choices of $\psi(x,\theta)$ with $\varphi(\theta) < 0$, solutions to (2) can become multivalued. Then trajectories appear to go backward in time, and the generating function (1), although locally inducing a canonical transformation, globally does not on the standard phase space. One may consider three "regularizations," without affecting the large-scale physics, all implementing a reduction of effective space. (i) Choose any branch, e.g., follow one branch until it disappears, then jump to another branch [Fig. 1(b)]. This is similar to the flea gas [60] (we do not know if there exist choices of $\psi(x,\theta)$ and branch that would exactly reproduce the flea gas algorithm). But, like for the flea gas, this regularization is not Hamiltonian or time reversible. (ii) Using the "hard-core" picture [39], relabel particles at the first collision [Fig. 1(c)]. This gives a time-reversible dynamics (no longer a tracer dynamics), as long as such collisions always appear before any time-backward parts of trajectories; this relabeling gives the rods in the hard-rod case. (iii) Inspired by Feynman's picture, interpret time-backward parts of trajectories as antiparticles [Fig. 1(d)]. The proximity of a particle (say orange in the figure) occasions a spontaneous particle-antiparticle pair creation (blue and green); the antiparticle (green) later annihilates with the incoming particle (blue), leaving the created particle (also blue) as outgoing physical particle. This is time-symmetric, and we believe it might define a canonical flow on the "Fock phase space" $\mathcal{F} = \bigoplus_g \mathbb{R}^{2Ng}$, which admits an arbitrary number g of solutions to (2); however, this would need to be investigated further. For smooth $\psi(x,\theta)$, multivaluedness can always be interpreted in this way, as the Eqs. (2), (3), for $\boldsymbol{x}$, $\boldsymbol{p}$, t, define smooth curves in $\mathbb{R}^{2N+1}$. We now argue that this picture naturally arises from $T\bar{T}$ deformations.

$T\bar{T}$ deformations.—Generalized $T\bar{T}$ deformations, as proposed in [43], are obtained as flows of Hamiltonians $H^{(\lambda)} \to H^{(\lambda)} + \delta H^{(\lambda)}$ parametrized by λ,

$$\delta H^{(\lambda)} = \delta\lambda \int d\theta d\alpha dx dx'\, w(x - x', \theta - \alpha) \times (q_\theta^{(\lambda)}(x) j_\alpha^{(\lambda)}(x') - j_\theta^{(\lambda)}(x) q_\alpha^{(\lambda)}(x')), \qquad (5)$$

with $w(x,\theta)$ some deformation function. Here, $q_\theta^{(\lambda)}(x)$ and $j_\theta^{(\lambda)}(x)$ are the charge densities and currents, with continuity equation $\partial_t q_\theta^{(\lambda)} + \partial_x j_\theta^{(\lambda)} = 0$, associated to the charge $Q_\theta^{(\lambda)} = \sum_i \delta(\theta - \theta_i^{(\lambda)})$ that measures the density of asymptotic momenta at θ in the deformed system. It turns out that if H is an integrable tracer dynamics, then so is $H^{(\lambda)}$, and $q_\theta^{(\lambda)}(x)$ and $j_\theta^{(\lambda)}(x)$ exist and are short-range. Remarkably, starting from a system of free particles $H^{(0)} = \sum_i p_i^2/2$, the deformed Hamiltonian $H^{(\lambda)}$ is nothing else but $H^{[\psi^\lambda]}$, Eq. (4), with $\psi^\lambda(x,\theta) = \lambda \int_{-\infty}^{\infty} dx' \,\mathrm{sgn}(x-x') w(x',\theta)$. This generalizes the mass-momentum deformation yielding hard rods [39]; see Ref. [55]. The semiclassical Bethe systems are the first concrete example of generalized $T\bar{T}$ deformations. We have rigorous proofs of these statements [53] under conditions guaranteeing invertibility of (2), (3), where $H^{[\psi^\lambda]}$ can be constructed on the standard phase space.

Going further, here we make the crucial observation that the relation Eq. (2), be it invertible or not, is still the correct $T\bar{T}$ deformation of the impact-parameter-position relation $y_i = x_i$. Indeed, $T\bar{T}$ deformations (5) can be obtained as *canonical flows* [35,39,53,61], and Eq. (2) arises directly from applying this flow. Thus, *multivaluedness may appear along the $T\bar{T}$ flow*, and pair creation and annihilation processes occur [Fig. 1(d)], as claimed; see Ref. [55]. In all cases, the dynamics remains local.

Thermodynamics.—Let us consider the generalized Gibbs ensembles [62], with Boltzmann weights $e^{-\sum_a \beta_a Q_a}$. We take more generally (x,θ)-dependent Lagrange parameters varying on scale L, with $e^{-\int dx d\theta\, \beta(x/L,\theta) q_\theta(x)} = e^{-\sum_i \beta(x_i/L,\theta_i)}$. In [53] we show, using methods of graph theory [63] and under certain further assumptions, that the free energy density

$$f_L = -\frac{1}{L}\log\sum_{N=0}^{\infty}\int\frac{d^N x d^N p}{N!} e^{-\sum_i \beta(x_i/L,\theta_i(\boldsymbol{x},\boldsymbol{p}))} \qquad (6)$$

is finite and given by $f_L = -(2\pi L)^{-1}\int dx d\theta\, e^{-\varepsilon_L(x,\theta)}$, where the pseudo energy $\varepsilon_L(x,\theta)$ satisfies

$$\varepsilon_L(x,\theta) = \beta\left(\frac{x}{L},\theta\right) - \int\frac{dx' d\theta'}{2\pi}\partial_\theta\partial_x\psi(x-x',\theta-\theta') e^{-\varepsilon_L(x',\theta')}. \qquad (7)$$

This is a TBA-like equation for Maxwell-Boltzmann statistics (see, e.g., [50]). The TBA is well known from quantum [64–66] and classical [9,10,12,14,15] integrability, at infinite volumes. It is striking that even at *finite volumes* the free energy possesses a TBA structure; the only result we are aware about this is for (positive-length) hard rods [52,67]. We postulate that the finite-volume free energy in interacting quantum and classical integrable systems is given by Eq. (7) under an appropriate choice of $\partial_\theta\partial_x\psi(x,\theta)$ reproducing the scattering shift $\varphi(\theta)$.

The infinite-volume limit of Eq. (7) yields the expected TBA equation $\varepsilon(\bar{x},\theta) = \lim_{L\to\infty}\varepsilon_L(L\bar{x},\theta) = \beta(\bar{x},\theta) - \int(d\theta'/2\pi)\varphi(\theta-\theta')e^{-\varepsilon(\bar{x},\theta')}$, from which $\lim_{L\to\infty} f_L$ follows. Thus, the thermodynamics of the semiclassical Bethe systems is described by the standard machinery of TBA, including its "local density approximation."

The free energy gives thermodynamic averages and fluctuations of conserved quantities. Interestingly, we also have the exact thermodynamic average $\rho_{\rm phys}(p) = n(\theta(p))/(2\pi)$ for the *physical momentum distribution* $L^{-1}\sum_i \delta(p-p_i)$. Here, $n(\theta) = e^{-\varepsilon(\theta)}$ is the occupation function and $\theta(p)$ is the inverse of the "dressed momentum" function $p^{\rm dr}(\theta)$. The latter is known to be the physical momentum of an excitation at Bethe root θ in Bethe ansatz systems, and is fixed by TBA equations; see Ref. [55]. As physical momenta of particles change throughout their trajectories, $\rho_{\rm phys}(p)$ is a quantity that is typically hard to access in integrable models; this is the first exact expression that we are aware of.

GHD.—We provide a heuristic argument for GHD to emerge in the hydrodynamic limit, paralleling Ref. [26]; other techniques [20] should give rigorous results.

We take macroscopic space and time, $x = L\bar{x}$, $t = L\bar{t}$ ($\bar{x}$, $\bar{t}$ finite, $L\to\infty$), with scaled coordinates $\bar{x}_i(\bar{t}) = x_i(t)/L$ and $\bar{y}_i = y_i/L$. The empirical density, $\rho_{\rm e}(\theta,\bar{x},\bar{t}) = L^{-1}\sum_i \delta(\bar{x}-\bar{x}_i(\bar{t}))\delta(\theta-\theta_i)$, is assumed to converge "weakly": $\rho_{\rm e}(\theta,\bar{x},\bar{t}) \to \rho_{\rm p}(\theta,\bar{x},\bar{t})$. Clearly,

$$\partial_{\bar{t}}\rho_{\rm e}(\theta,\bar{x},\bar{t}) + \partial_{\bar{x}}\left(L^{-1}\sum_i \dot{\bar{x}}_i\delta(\bar{x}-\bar{x}_i)\delta(\theta-\theta_i)\right) = 0. \qquad (8)$$

Rewriting Eq. (2) as $\bar{x}_i(\bar{t}) = \bar{y}_i + \theta_i\bar{t} - (1/L)\sum_{j\neq i}\partial_\theta\psi(L(\bar{x}_i(\bar{t})-\bar{x}_j(\bar{t})),\theta_i-\theta_j)$, we assume that, for every i, there is a fraction of particles j that tend to 1 as $L\to\infty$ such that $\bar{x}_i(\bar{t}) - \bar{x}_j(\bar{t}) > 0$. Then, we can replace $\partial_\theta\psi(L\bar{x},\theta) \to \psi^{\mathrm{sgn}(\bar{x})}\varphi(\theta)$, where $\psi^+ - \psi^- = 1$. Taking the $\bar{t}$ derivative, we find

$$\dot{\bar{x}}_i = \theta_i - \frac{1}{L}\sum_{j\neq i}\delta(\bar{x}_i-\bar{x}_j)\varphi(\theta_i-\theta_j)(\dot{\bar{x}}_i-\dot{\bar{x}}_j) \quad (L\to\infty). \qquad (9)$$

Making the ansatz $\dot{\bar{x}}_i = f(\bar{x}_i,\theta_i)$, the second term on the right-hand side is $-\int d\theta\rho_{\rm e}(\theta,\bar{x}_i,\bar{t})\varphi(\theta_i-\theta)[f(\bar{x}_i,\theta_i) - f(\bar{x}_i,\theta)]$. Thus, $f(\bar{x},\theta) = v^{\rm eff}_{[\rho_{\rm e}(\cdot,\bar{x},\bar{t})]}(\theta)$ solves

$$v^{\rm eff}_{[\rho]}(\theta) = \theta - \int d\theta'\rho(\theta')\varphi(\theta-\theta')[v^{\rm eff}_{[\rho]}(\theta) - v^{\rm eff}_{[\rho]}(\theta')]. \qquad (10)$$

This is exactly the equation for the effective velocity in GHD [1,2]. Putting this into Eq. (8) and taking the limit $L\to\infty$ we obtain the GHD equation,

$$\partial_{\bar{t}}\rho_{\rm p}(\theta,\bar{x},\bar{t}) + \partial_{\bar{x}}(v^{\rm eff}_{[\rho_{\rm p}]}(\theta,\bar{x},\bar{t})\rho_{\rm p}(\theta,\bar{x},\bar{t})) = 0. \qquad (11)$$

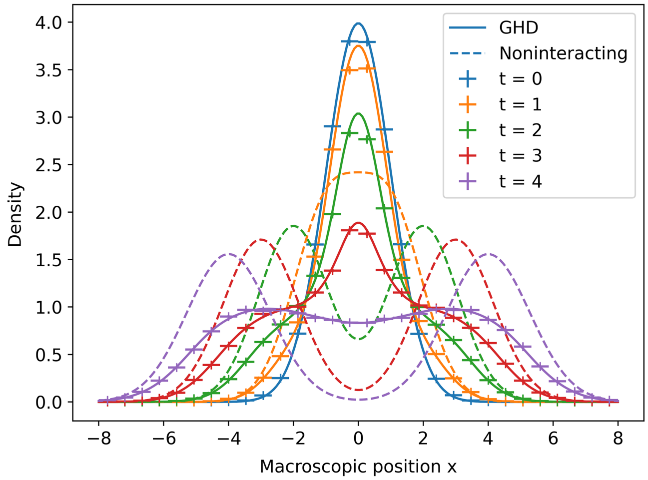

FIG. 2. Evolution of the microscopic particle density (data points) and that from the GHD solution (solid lines) for different times. The initial state is $\rho_{\rm p}(\theta,x,0) = (25/2\pi)e^{-x^2/2}(e^{-25(\theta-1)^2/2} + e^{-25(\theta+1)^2/2})$. We use $\psi(x,\theta) = (x/2\sqrt{x^2+\alpha^2})\phi(\theta)$, where $\alpha = 0.1$ and $\phi(\theta) = 2\arctan(\theta/c)$. The microscopic results were averaged over 100 samples, each starting from a randomly generated initial state ($L = 300$, each sample contains $N \approx 3000$ particles). The error bars indicate the standard deviation and the bin size used for computing the density of particles. We also give the density for noninteracting particles (dashed lines) for comparison.

Thus, we have shown that, at large scales, the semiclassical Bethe system satisfies the GHD equation. This is not rigorous—in particular, in Eq. (9) one would need to use a regularization of the delta function. We give an alternative derivation of GHD in [55], based on the fact that the metric change $dy_i = K_i(x_i)dx_i$ converges to the GHD change of metric $K_i(x) \to 2\pi\rho_{\rm s}(\theta_i,x)$ determined by the "space" or "total" density $\rho_{\rm s}(\theta,x)$ [49].

We numerically demonstrate that the GHD equation correctly captures the large-scale behavior in an explicit example; see Fig. 2. For illustration, we use the phase shift from the quantum Lieb-Liniger model, but with an initial state that breaks the maximal fermionic occupation allowed by quantum mechanics (the maximal density of particle per state is $6.264 > 1$). This initial state is nevertheless realizable, and its hydrodynamics makes sense, as indeed it is realized by a semiclassical Bethe system. The details of the numerical simulations can be found in [55]. Compared to the evolution of noninteracting particles the expansion of the interacting particles is much slower, which is in line with the intuitive meaning of a positive phase shift as an effective time delay during the scattering of two particles.

Conclusions.—We introduced a new class of classical integrable models, dubbed semiclassical Bethe systems for their relation with the quantum Bethe ansatz, obtained as generalized $T\bar{T}$ deformations of classical noninteracting particles. In these systems, each particle is a "tracer" that has the same incoming and outgoing momentum. The class is parametrized by a function determining the microscopic dynamics, and displays factorized scattering with a *(largely) arbitrary two-body shift*, including those found in many quantum integrable models. The microscopic dynamics displays special features, including pair creations and annihilations; the thermodynamics in finite volumes surprisingly takes a form akin to the thermodynamic Bethe ansatz (TBA), reducing to the TBA at infinite volume; the distribution in physical phase space can be evaluated exactly; and the large-scale dynamics is described by GHD, and therefore *identical* to that of any quantum or classical integrable system with the same chosen two-body shift. We conjecture that, with short-range interaction, the agreement persists at higher orders: the models should encode the universal hydrodynamic expansion of classical many-body integrability, as corrections due to specific interactions should be exponentially subleading. For instance, particles' positions in our models should approximate well the spatial distribution of solitons in dense soliton gases, something that can be useful for initial state preparation.

It would be interesting to construct the full particle-non-conserving Hamiltonian description of trajectories (2), (3) with negative shifts $\varphi(\theta) < 0$ [Fig. 1(d)]. Finding the full integrability structure of our models, perhaps connecting with sine-Gordon soliton trajectories [68–70], would be interesting, as would quantizing our models, perhaps in the spirit of [26] (integrability of generalized $T\bar{T}$-deformed systems is established [43]); the notion of pair creation and annihilation may play an important role. Finally, adding an external potential is possible, and we anticipate that rigorous proofs of the emergence of the GHD equation can be obtained following ideas in the hard-rod case [8,20]. This might also shed light on GHD beyond the Euler scale.

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